DENSE FUNCTORS AND DENSITY PRESENTATIONS

Research Frontiers 2

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Abstract. Dense functors in ordinary and additive categories have been considered especially by Isbell [Isb60], Ulmer [Ulm68] and Diers [Die76]. In the more general context of enriched categories they were treated by Kelly [Kel82] and Day [Day74]; moreover in [Day77] the notion of density presentation was defined and developed. After introducing some notations and our background settings, we outline in sections 2 and 3 the most important features of density presentation following both [Day77] and [Kel82]. In the remaining sections we apply the preceding results in the context of monads, free cocompletions and reflective and coreflective subcategories.

Contents

1. Notations and Background

We fix here a complete and cocomplete symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ as our base for enrichment. All notions should be understood as $\mathcal{V}$-enriched, so for example by saying category and functor we really mean $\mathcal{V}$-category and $\mathcal{V}$-functor.

We also allow all our categories to be large, unless specified otherwise. An unfortunate consequence of this is that, given $\mathcal{A}$, the presheaf category $[\mathcal{A}^{op}, \mathcal{V}]$ may not exist as $\mathcal{V}$-enriched category. We can avoid this problem considering $[\mathcal{A}^{op}, \mathcal{V}]$ as a $\mathcal{V}'$-category for some extension $\mathcal{V}'$ of $\mathcal{V}$ (see [Kel82]); this allows us to still work with them.

We shall often use the notion of weighted colimit (and limit): given functors $\varphi : \mathcal{K}^{op} \to \mathcal{V}$ and $F : \mathcal{K} \to \mathcal{C}$, the colimit of F weighted by φ, if it exists in $\mathcal{C}$, is denoted by $\varphi * F$ and satisfies

$$\mathcal{C}(\varphi * F, c) \cong [\mathcal{K}^{op}, \mathcal{V}](\varphi, \mathcal{C}(F-, c))$$

naturally in $c \in \mathcal{C}$. When $\mathcal{K}$ is not small (as is possible in our setting) this means that $[\mathcal{K}^{op}, \mathcal{V}](\varphi, \mathcal{C}(F-, c))$ exists as an element of $\mathcal{V}$ for each $c \in \mathcal{C}$ and $\varphi * F$ satisfies the above condition.

Given a category $\mathcal{C}$, we denote by $|\mathcal{C}|$ the discrete category on $\mathcal{C}$; this has the same objects of $\mathcal{C}$ but its hom-objects are $\mathcal{C}(c, d) := 0$ the initial object of $\mathcal{V}_0$ for each $c \neq d \in \mathcal{C}$, and $\mathcal{C}(c, c) := I$.

More generally, given an ordinary locally small category $\mathcal{L}$, we'll consider the free $\mathcal{V}$-category $\mathcal{L}_{\mathcal{V}}$ over $\mathcal{L}$; it has the same objects of $\mathcal{L}$ but hom-objects given by $\mathcal{L}_{\mathcal{V}}(l, m) := \mathcal{L}(l, m) \cdot I$ the coproduct of $\mathcal{L}(l, m)$ copies of I in $\mathcal{V}_0$.

Since our main topic will be dense functors, we shall make use of the following notations: for any functor $N : \mathcal{A} \to \mathcal{C}$ denote by $\widetilde{N}$ the composite:

$$\begin{array}{rccccc} \widetilde{N} : \mathcal{C} & \xrightarrow{\;Y\;} & [\mathcal{C}^{op}, \mathcal{V}] & \xrightarrow{[N^{op}, 1]} & [\mathcal{A}^{op}, \mathcal{V}] \\ c & \dashrightarrow & \mathcal{C}(-, c) & \dashrightarrow & \mathcal{C}(N-, c) \end{array}$$

and, for $J : \mathcal{K} \to \mathcal{A}$:

$$\begin{array}{rccccc} \widehat{J} : \mathcal{A}^{op} & \xrightarrow{\;Y'\;} & [\mathcal{A}, \mathcal{V}] & \xrightarrow{[J, 1]} & [\mathcal{K}, \mathcal{V}] \\ a & \dashrightarrow & \mathcal{A}(a, -) & \dashrightarrow & \mathcal{A}(a, J-) \end{array}$$

where Y and Y' are respectively the covariant and controvariant Yoneda embeddings. These are also known as $\mathcal{C}(N, 1)$ and $\mathcal{A}(1, J)$ respectively (see [SW78] for instance), but we prefer the more compact notation $\widetilde{N}$ and $\widehat{J}$.

Next we recall the definition and some of the main properties of left Kan extensions.

Definition 1.1 ([Kel82])**.** Given $N : \mathcal{A} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$, the left Kan extension of F along N is a functor $G : \mathcal{C} \to \mathcal{B}$ together with a natural transformation $\phi : F \to GN$ such that for each $H : \mathcal{C}^{op} \to \mathcal{V}$, ϕ induces an isomorphism

$$HN^{op} * F \longrightarrow H * G.$$

We denote the left Kan extension of F along N by $\mathrm{Lan}_N F$; it is easy to see that this exists iff the colimits $\widetilde{N}c * F$ exist in $\mathcal{B}$ for each $c \in \mathcal{C}$, and in that case is given by $\mathrm{Lan}_N F = \widetilde{N}(-) * F$. The following universal property is a consequence of definitions:

Theorem 1.2. *Given $N : \mathcal{A} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$, if $(\mathrm{Lan}_N F, \phi)$ exists then ϕ induces a natural isomorphism*

$$[\mathcal{C}, \mathcal{B}](\mathrm{Lan}_N F, S) \cong [\mathcal{A}, \mathcal{B}](F, SN)$$

for any $S : \mathcal{C} \to \mathcal{B}$.

2. Dense Functors and Presentations

We start this section with the definition of dense functor; this is probably the easiest one but at the same time the least intuitive. Other equivalent ways of defining them will be given later.

Definition 2.1. A functor $N : \mathcal{A} \to \mathcal{C}$ is called **dense** if $\widetilde{N} : \mathcal{C} \to [\mathcal{A}^{op}, \mathcal{V}]$ is fully faithful.

Now, given categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, denote by $\mathcal{A}$-Coct$[\mathcal{C}, \mathcal{B}]$ the full subcategory of $[\mathcal{C}, \mathcal{B}]$ of those functors preserving all existing colimits $\varphi * G$ with weight $\varphi : \mathcal{A}^{op} \to \mathcal{V}$.

The following Proposition gives a first characterization of dense functors and also a more intuitive way of thinking about them.

Proposition 2.2. *Given* $N : \mathcal{A} \to \mathcal{C}$*, the following facts are equivalent:*

(1) N *is dense;*
(2) *for each* $\mathcal{B}$*, precomposition with* N *induces a fully faithful functor*
$$[N, 1] : \mathcal{A}\text{-Coct}[\mathcal{C}, \mathcal{B}] \to [\mathcal{A}, \mathcal{B}];$$
(3) id_N *exhibits* $1_{\mathcal{C}}$ *as* $\mathrm{Lan}_N N$.

Proof. (2) ⇒ (1). Consider the Yoneda embedding $Y : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}] \simeq [\mathcal{C}, \mathcal{V}^{op}]^{op}$; since representables are continuous, Y lands in $(\mathcal{A}\text{-Coct}[\mathcal{C}, \mathcal{V}^{op}])^{op}$. To conclude then note that $\widetilde{N} = [N, 1]^{op} \circ Y$; thus, by (2), it is full and faithful.

(1) ⇒ (3). By hypothesis for each c and d in $\mathcal{C}$,
$$\widetilde{N}_{cd} : \mathcal{C}(c, d) \longrightarrow [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \widetilde{N}d) \cong \mathcal{C}(\widetilde{N}c * N, d)$$
is invertible. As a consequence $c \cong \widetilde{N}c * N$, where the isomorphism is induced by the identity map $id_{\widetilde{N}c} : \widetilde{N}c \to \widetilde{N}c$. Then (3) follows.

(3) ⇒ (2). Let $\mathcal{B}$ be any category and $F : \mathcal{C} \to \mathcal{B}$ an $\mathcal{A}$-cocontinuous functor; then $F(\widetilde{N}c * N) \cong \widetilde{N}c * FN$ for each c in $\mathcal{C}$, i.e. $\mathrm{Lan}_N(FN) \cong F(\mathrm{Lan}_N N) \cong F$. Then by the universal property 1.2 of the left Kan extension we get for each $S : \mathcal{C} \to \mathcal{B}$:
$$[\mathcal{C}, \mathcal{B}](F, S) \cong [\mathcal{C}, \mathcal{B}](\mathrm{Lan}_N(FN), S) \cong [\mathcal{A}, \mathcal{B}](FN, SN)$$
the isomorphism being given by $[N, 1]_{FS}$. It follows then that the restriction of $[N, 1]$ to $\mathcal{A}$-Coct$[\mathcal{C}, \mathcal{B}]$ is fully faithful. □

If N is fully faithful, condition (2) says exactly that a full subcategory $\mathcal{A}$ of $\mathcal{C}$ is dense iff $\mathcal{A}$-cocontinuous functors with domain $\mathcal{C}$ are determined by their restriction to $\mathcal{A}$; this evidently calls to mind the more common notion of topological density between subspaces.

Dense functors which are also fully faithful can be characterized this way:

Proposition 2.3. *A category $\mathcal{C}$ is equivalent to a full subcategory of a presheaf category $[\mathcal{A}^{op}, \mathcal{V}]$ containing the representables iff there is a fully faithful and dense functor $N : \mathcal{A} \to \mathcal{C}$.*

Proof. On one side, let $J : \mathcal{C} \to [\mathcal{A}^{op}, \mathcal{V}]$ be the inclusion; then, since $\mathcal{C}$ contains the representables, we can consider as N the codomain restriction of the Yoneda embedding $Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathcal{V}]$. It's easy to see that $J \cong \widetilde{N}$, hence N is dense and fully faithful.
On the other hand, if N is fully faithful and dense then $\widetilde{N}N \cong Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathcal{V}]$ so that $\mathcal{C}$ contains the representables. □

Proving that a given functor is dense using one of the three equivalent conditions above may be rather hard; this is where the notion of density presentation comes to help. The following definition was introduced and developed by Day in [Day77].

Definition 2.4. A **presentation** for $N : \mathcal{A} \to \mathcal{C}$ is given by a quadruple $(\mathcal{K}, J, \varphi, \xi)$ with elements: a category $\mathcal{K}$, an index functor $J : \mathcal{K} \to \mathcal{A}$, a coefficient functor $\varphi : \mathcal{K}^{op} \otimes |\mathcal{C}| \to \mathcal{V}$ such that $\varphi(-, c) * NJ$ exists in $\mathcal{C}$ for each c, and a morphism $\xi_c : \varphi(-, c) * NJ \to c$ for each $c \in \mathcal{C}$.

Recall that $|\mathcal{C}|$ is the discrete category on $\mathcal{C}$; thus, the coefficient functor φ is just a collection of functors $\varphi_c := \varphi(-, c) : \mathcal{K}^{op} \to \mathcal{V}$ for each object c of $\mathcal{C}$.

Given a presentation $\mathcal{P} = (\mathcal{K}, J, \varphi, \xi)$; for each $c \in \mathcal{C}$ and $a \in \mathcal{A}$ there is an induced map $\bar{\xi}_{c,a} : \varphi_c * \widehat{J}a \to \mathcal{C}(Na, \varphi_c * NJ)$ in $\mathcal{V}$ defined as the composite:

$$\varphi_c * \widehat{J}a = \varphi_c * \mathcal{A}(a, J-) \xrightarrow{\varphi_c * N} \varphi_c * \mathcal{C}(Na, NJ-) \xrightarrow{can} \mathcal{C}(Na, \varphi_c * NJ);$$

where *can* denotes the canonical comparison expressing the extent to which $\mathcal{C}(Na, -)$ preserves the colimit $\varphi_c * NJ$. For each c in $\mathcal{C}$, this gives a natural transformation $\bar{\xi}_c : \varphi_c * \widehat{J}(-) \to \widetilde{N}(\varphi_c * NJ)$ in $[\mathcal{A}^{op}, \mathcal{V}]$.

Note moreover that $\mathcal{P}$ induces the following isomorphisms in $\mathcal{V}$ for each $c, d \in \mathcal{C}$:

$$\begin{aligned} \mathcal{C}(\varphi_c * NJ, d) &\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, \mathcal{C}(NJ-, d)) \\ &\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c -, [\mathcal{A}^{op}, \mathcal{V}](\mathcal{A}(\Box, J-), \mathcal{C}(N\Box, d))) \\ &\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, [\mathcal{A}^{op}, \mathcal{V}](\widehat{J}, \widetilde{N}d)) \\ &\cong [\mathcal{A}^{op}, \mathcal{V}](\varphi_c * \widehat{J}, \widetilde{N}d) \end{aligned} \tag{1}$$

where recall that $\widetilde{N} = \mathcal{C}(N, -)$ and $\widehat{J} = \mathcal{A}(-, J)$. We also point out that the composite of all the isomorphisms in (1) is given by

$$\mathcal{C}(\varphi_c * NJ, d) \xrightarrow{\widetilde{N}} [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_c * NJ), \widetilde{N}d) \xrightarrow{[\mathcal{A}^{op}, \mathcal{V}](\bar{\xi}_c, \widetilde{N}d)} [\mathcal{A}^{op}, \mathcal{V}](\varphi_c * \widehat{J}, \widetilde{N}d)$$

Of course, to get information about the density of N, a presentation $\mathcal{P} = (\mathcal{K}, J, \varphi, \xi)$ like in the previous definition is not enough. The existence of such a $\mathcal{P}$ only guarantees that for each $c \in \mathcal{C}$ a certain colimit with values in the image of $\mathcal{A}$ exists in $\mathcal{C}$ and is in some way connected to the object c through ξ_c and $\bar{\xi}_c$.

In order to obtain the density of N from such a presentation we need to ask for some properties of these connecting morphisms ξ_c and $\bar{\xi}_c$:

Definition 2.5. Let $(\mathcal{K}, J, \varphi, \xi)$ be a presentation for $N : \mathcal{A} \to \mathcal{C}$; we say that it is a:

- **generating** presentation if ξ_c is an epimorphism for each $c \in \mathcal{C}$;
- **density** presentation if ξ_c is a regular epimorphism, and $\mathcal{C}(Na, \xi_c)$ and $\bar{\xi}_{c,a}$ are epimorphisms for each $c \in \mathcal{C}$ and $a \in \mathcal{A}$;
- **strict** presentation if both ξ_c and $\bar{\xi}_{c,a}$ are isomorphisms for each $c \in \mathcal{C}$ and $a \in \mathcal{A}$.

A direct consequence of this definition is the following result:

Proposition 2.6. *Let $\mathcal{P} := (\mathcal{K}, J, \varphi, \xi)$ be a presentation for N; then:*

(a) *if $\mathcal{P}$ is generating, then $\widetilde{N}$ is faithful;*
(b) *if $\mathcal{P}$ a density presentation, then N is dense;*
(c) *if $\mathcal{P}$ is strict, then N is dense and for each $c \in \mathcal{C}$ and $G : \mathcal{A}^{op} \to \mathcal{V}$,*

$$[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, G) \cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, GJ).$$

Proof. (a). For each c, d in $\mathcal{C}$ consider the commutative diagram:

$$\begin{tikzcd}
\mathcal{C}(c,d) \arrow[rr, "\mathcal{C}(\xi_c, d)"] \arrow[d, "\widetilde{N}_{cd}"'] & & \mathcal{C}(\varphi_c * NJ, d) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\widetilde{N}"] & {[\mathcal{A}^{op}, \mathcal{V}](\varphi_c * \widehat{J}, \widetilde{N}d)} \\
{[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \widetilde{N}d)} \arrow[rr, "{[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}\xi_c, \widetilde{N}d)}"'] & & {[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_c * NJ), \widetilde{N}d)} \arrow[ur, "{[\mathcal{A}^{op}, \mathcal{V}](\bar{\xi}_c, \widetilde{N}d)}"'] &
\end{tikzcd}$$

(the triangle commutes by 1). Since ξ_c is by assumption an epimorphism, $\mathcal{C}(\xi_c, d)$ is a monomorphism. As a consequence $\widetilde{N}_{cd}$ is a monomorphism too and hence N is faithful.

(b). Suppose now that $(\mathcal{K}, J, \varphi, \xi)$ is a density presentation. Consider for each $b, d \in \mathcal{C}$ the isomorphism

$$\mathcal{C}(\varphi_b * NJ, d) \xrightarrow{\widetilde{N}} [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_b * NJ), \widetilde{N}d) \xrightarrow{[\mathcal{A}^{op}, \mathcal{V}](\bar{\xi}_b, \widetilde{N}d)} [\mathcal{A}^{op}, \mathcal{V}](\varphi_b * \widehat{J}, \widetilde{N}d)$$

then $[\mathcal{A}^{op}, \mathcal{V}](\bar{\xi}_b, \widetilde{N}d)$ is a split epimorphism, but it also is a monomorphism (since $\bar{\xi}_b$ is epi) and hence is invertible. As a consequence $\widetilde{N} : \mathcal{C}(\varphi_b * NJ, d) \to [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_b * NJ), \widetilde{N}d)$ is itself an isomorphism.

Now fix c, d in $\mathcal{C}$ as before; by hypothesis ξ_c is a regular epimorphism, i.e. is the coequalizer of two maps $u, v : e \to (\varphi_c * NJ)$. Precomposing u and v with ξ_e (which is also a regular epimorphism) we get ξ_c as the coequalizer of $u \circ \xi_e, v \circ \xi_e : \varphi_e * NJ \to \varphi_c * NJ$. Consider now the previous diagram completed with u, v and ξ_e:

$$\begin{tikzcd}
\mathcal{C}(c,d) \arrow[r, "\mathcal{C}(\xi_c, d)"] \arrow[d, "\widetilde{N}_{cd}"'] & \mathcal{C}(\varphi_c * NJ, d) \arrow[r, shift left, "\mathcal{C}(u \circ \xi_e, d)"] \arrow[r, shift right, "\mathcal{C}(v \circ \xi_e, d)"'] \arrow[d, "\cong"] & \mathcal{C}(\varphi_e * NJ, d) \arrow[d, "\cong"] \\
{[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \widetilde{N}d)} \arrow[r, "m_{cd}"'] & {[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_c * NJ), \widetilde{N}d)} \arrow[r, shift left] \arrow[r, shift right] & {[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_e * NJ), \widetilde{N}d)}
\end{tikzcd}$$

where the two vertical arrows in the right square are isomorphisms by the previous argument and $m_{cd} = [A^{op}, \mathcal{V}](\widetilde{N}\xi_c, \widetilde{N}_d)$ is a monomorphism since by hypothesis $\widetilde{N}\xi_c$ is an epimorphism. Note moreover that $\mathcal{C}(\xi_c, d)$ is the equalizer of $\mathcal{C}(u \circ \xi_e, d)$ and $\mathcal{C}(v \circ \xi_e, d)$ since $\mathcal{C}(-, d)$ transforms colimits into limits; this and the commutativity of the diagram imply that there exists $r : [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \widetilde{N}d) \to \mathcal{C}(c, d)$ such that $\mathcal{C}(\xi_c, d) \circ r \cong m_{cd}$. Then, using that m_{cd} is monomorphism, it is easy to see that r is a right inverse of $\widetilde{N}_{cd}$. Hence $\widetilde{N}_{cd}$ is a split epimorphism; by (a) it is also a monomorphism and hence is invertible; this means exactly that N is dense.

(c). Fix c in $\mathcal{C}$ and $G : \mathcal{A}^{op} \to \mathcal{V}$; then:

$$\begin{aligned}
[\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, G) &\overset{\xi_c}{\cong} [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}(\varphi_c * NJ), G) \\
&\overset{\bar{\xi}_c}{\cong} [\mathcal{A}^{op}, \mathcal{V}](\varphi_c * \widehat{J}, G) \\
&\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, [\mathcal{A}^{op}, \mathcal{V}](\widehat{J}-, G)) \\
&\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, GJ).
\end{aligned}$$

□

When N is fully faithful we can also consider another notion of presentation which appears for example in Section 5 of [Kel82].

Definition 2.7. Let $N : \mathcal{A} \to \mathcal{C}$ be fully faithful; a **discrete** presentation for N is a (not necessarily small) family $(\mathcal{K}_\gamma,\ F_\gamma : \mathcal{K}_\gamma^{op} \to \mathcal{V};\ P_\gamma : \mathcal{K}_\gamma \to \mathcal{A})_{\gamma \in \Gamma}$ such that the colimit $F_\gamma * NP_\gamma$ exists in $\mathcal{C}$ for each γ and is N-absolute (i.e. is preserved by $\widetilde{N}$), and $\mathcal{C}$ is the closure of $\mathcal{A}$ under this family of colimits.

Note that, given a strict presentation of a fully faithful N, the fact that ξ_c and $\bar{\xi}_c$ are invertible says exactly that the colimits $\varphi_c * NJ$ are preserved by $\widetilde{N}$ and hence are N-absolute.

The following Theorem connects Day's notion of presentation with that just introduced by proving that when N is a full inclusion there is no difference between them.

Theorem 2.8. *Given $N : \mathcal{A} \to \mathcal{C}$, the following are equivalent:*

(1) *N is dense;*
(2) *N has a strict presentation.*

If moreover N is fully faithful, they are further equivalent to:

(3) *N has a discrete presentation.*

Proof. (1) $\Rightarrow$ (2). It is enough to consider the presentation $\mathcal{P} = (\mathcal{K}, J, \varphi, \xi)$ with $\mathcal{K} := \mathcal{A}$, $J := id_{\mathcal{A}}$, $\varphi_c := \widetilde{N}c$ and $\xi_c := \widetilde{N}c * N \xrightarrow{\cong} c$. Then $\bar{\xi}_{ca}$ is also invertible since

$$\varphi_c * \mathcal{A}(a, J-) = \widetilde{N}c * \mathcal{A}(a, -) \cong \mathcal{C}(Na, c) \cong \mathcal{C}(Na, \widetilde{N}c * N) = \mathcal{C}(Na, \varphi_c * NJ).$$

Thus $\mathcal{P}$ is a strict presentation.

(2) $\Rightarrow$ (1). Follows directly from the preceding Theorem.

(2) $\Rightarrow$ (3). Given $\mathcal{P} = (\mathcal{K}, J, \varphi, \xi)$ strict, it is enough to consider the family

$$(\mathcal{K}_c := \mathcal{K}; F_c := \varphi_c; P_c := J)_{c\in\mathcal{C}}.$$

Indeed, the closure of $\mathcal{A}$ under $F_c * NP_c$ is $\mathcal{C}$ since $c \cong \varphi_c * NJ = F_c * NP_c$ for each c ($\mathcal{P}$ is strict); moreover these colimits are N-absolute:

$$\widetilde{N}(\varphi_c * NJ) = \mathcal{C}(N-, \varphi_c * NJ) \overset{\bar{\xi}_c}{\cong} \varphi_c * \widehat{J} \cong \varphi_c * \widetilde{N}(NJ),$$

where the last holds since N is fully faithful.

(3) $\Rightarrow$ (2). Consider a family as in (3); then for each $c \in \mathcal{C}$ there exists $\gamma_c \in \Gamma$ such that $c \cong F_{\gamma_c} * NP_{\gamma_c}$ $(*_1)$ and $\widetilde{N}(F_{\gamma_c} * NP_{\gamma_c}) \cong F_{\gamma_c} * \widetilde{N}NP_{\gamma_c}$ $(*_2)$. Define a strict presentation $\mathcal{P} = (\mathcal{K}, J, \varphi, \xi_c)$ by:

$$\mathcal{K} := \sum_{c\in\mathcal{C}} \mathcal{K}_{\gamma_c},$$

$$J := (P_{\gamma_c})_{c\in\mathcal{C}} : \mathcal{K} \to \mathcal{A}$$

and $\varphi_c := (\psi_d)_d : \mathcal{K}^{op} \to \mathcal{V}$, where $\psi_d : \mathcal{K}^{op}_{\gamma_d} \to \mathcal{V}$ are defined as

$$\psi_d = \begin{cases} F_{\gamma_c} & d = c \\ 0 & d \neq c \end{cases}$$

Condition $(*_1)$ defines ξ_c and together with $(*_2)$ assures that the presentation $\mathcal{P}$ so defined is strict. $\square$

3. Left Kan Extensions and Adjoints

We are now going to see some properties implied by strict density presentations. Let us fix henceforth $N : \mathcal{A} \to \mathcal{C}$ (not necessarily fully faithful) and a strict presentation $\mathcal{P} = (\mathcal{K}, J, \varphi, \xi)$.

Proposition 3.1. *For each $G : \mathcal{A} \to \mathcal{B}$, if $\varphi_c * GJ$ exists in $\mathcal{B}$ then $\varphi_c * GJ \cong \widetilde{N}c * G$ for each $c \in \mathcal{C}$ and $\mathrm{Lan}_N G : \mathcal{C} \to \mathcal{B}$ exists.*

Proof. Let $Y : \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{V}]$ be the Yoneda embedding; since Y is fully faithful and transforms colimits of $\mathcal{B}$ into limits, it is enough to prove that $Y(\varphi_c * GJ) \cong \{\widetilde{N}c, YG\}$ (this implies that $\widetilde{N}c * G$ exists in $\mathcal{B}$ and is $\varphi_c * GJ$). Let then $b \in \mathcal{B}$:

$$\begin{aligned}
\{\widetilde{N}c, YG\}(b) &\cong \{\widetilde{N}c, \mathcal{B}(G-, b)\} \\
&\cong [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \mathcal{B}(G-, b)) \\
&\overset{2.6}{\cong} [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, \mathcal{B}(GJ-, b)) \\
&\cong \{\varphi_c, YGJ\}(b).
\end{aligned}$$

Hence $\{\widetilde{N}c, YG\} \cong \{\varphi_c, YGJ\} \cong Y(\varphi_c * GJ)$ as desired. About the existence of $\mathrm{Lan}_N G$, it follows since $\mathrm{Lan}_N G(c)$ is defined as $\widetilde{N}c * G$ and this exists by the previous argument. □

Corollary 3.2. *Given $F : \mathcal{C} \to \mathcal{B}$, id_{FN} exhibits F as $\mathrm{Lan}_N(FN)$ iff F preserves $\varphi_c * NJ$ for each c in $\mathcal{C}$.*

Proof. We have $Fc \cong F(\varphi_c * NJ)$ through $F\xi_c$ and $\mathrm{Lan}_N(FN) \cong \widetilde{N}c * FN \cong \varphi_c * FNJ$ by the previous Proposition applied to FN. It follows that $Fc \cong \mathrm{Lan}_N(FN)c$ iff F preserves $\varphi_c * NJ$. □

We are now going to see how strict presentations are related to adjoints.

Definition 3.3. Let $R : \mathcal{B} \to \mathcal{C}$, $N : \mathcal{A} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$, we say that F is a left N-adjoint of R (and R is a right N-adjoint of F), written $F \dashv_N R$, if for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$

$$\mathcal{B}(Fa, b) \cong \mathcal{C}(Na, Rb)$$

naturally in a and b.

Proposition 3.4. *Let $\mathcal{P}$ be a strict presentation for $N : \mathcal{A} \to \mathcal{B}$. Then:*

1. *$F : \mathcal{C} \to \mathcal{B}$ has a right adjoint iff it preserves $\varphi_c * NJ$ for each $c \in \mathcal{C}$ and FN has a right N-adjoint.*
2. *If $F \dashv_N R : \mathcal{B} \to \mathcal{C}$ and $\varphi_c * FJ$ exists in $\mathcal{B}$ for each c, then R has a left adjoint.*

Proof. (1). Suppose that F preserves $\varphi_c * NJ$ and $FN \dashv_N R : \mathcal{B} \to \mathcal{C}$. Then, by Corollary 3.2, $F \cong \mathrm{Lan}_N(FN)$, i.e. $Fc \cong \widetilde{N}c * FN$ for each c. As a consequence $F \dashv R$:

$$\begin{aligned}\mathcal{B}(Fc, b) &\cong \mathcal{B}(\widetilde{N}c * FN, b) \\ &\cong [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \mathcal{B}(FN-, b)) \\ &\cong [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \mathcal{C}(N-, Rb)) \quad (FN \dashv_N R) \\ &\cong \mathcal{C}(c, Rb) \quad (N \text{ dense})\end{aligned}$$

Vice versa, let $R : \mathcal{B} \to \mathcal{C}$ be a right adjoint for F; then F preserves all colimits and by adjointness $\mathcal{B}(FNa, b) \cong \mathcal{C}(Na, Rb)$. In particular F preserves $\varphi_c * NJ$ and $FN \dashv_N R$.

(2). Let $\bar{F} : \mathcal{C} \to \mathcal{B}$ be defined as $\bar{F}c := \varphi_c * FJ$; then:

$$\begin{aligned}\mathcal{B}(\bar{F}c, b) &= \mathcal{B}(\varphi_c * FJ, b) \\ &\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, \mathcal{B}(FJ-, b)) \\ &\cong [\mathcal{K}^{op}, \mathcal{V}](\varphi_c, \mathcal{C}(NJ-, Rb)) \quad (F \dashv_N R) \\ &\cong \mathcal{C}(\varphi_c * NJ, Rb) \cong \mathcal{C}(c, Rb).\end{aligned}$$

Note that a priori $\bar{F}$ is not well defined (since φ_c is discrete on $c \in \mathcal{C}$), but as a consequence of these isomorphisms it becomes an actual functor $\bar{F} : \mathcal{C} \to \mathcal{B}$. □

The following is the main theorem of this section, it explains in which circumstances an N-adjunction induces an equivalence of categories. We are going to see a direct application of this in the next section.

Theorem 3.5. *Let $\mathcal{P}$ be a strict presentation for $N : \mathcal{A} \to \mathcal{C}$. Suppose that $F \dashv_N R : \mathcal{B} \to \mathcal{C}$ and let $\bar{F} \dashv R$ be given with $\bar{F}c = \varphi_c * FJ$. Then:*

1. *If R is conservative, preserves $\varphi_{Rb} * FJ$ for each $b \in \mathcal{B}$, and $N \cong RF$, then R is fully faithful and F is dense.*
2. *R is an equivalence (with inverse $\bar{F}$) iff it is conservative, preserves $\varphi_c * FJ$ for each $c \in \mathcal{C}$ and $N \cong RF$.*

Proof. (1). By hypothesis $\bar{F} \dashv R$; thus for each $b \in \mathcal{B}$ we can consider the counit of the adjunction $\bar{F}Rb \to b$ which, together with R, induces the following isomorphisms:

$$\begin{aligned}R\bar{F}Rb &\cong R(\varphi_{Rb} * FJ) \quad (\textit{def of } \bar{F}) \\ &\cong \varphi_{Rb} * RFJ \quad (\textit{by } 1) \\ &\cong \varphi_{Rb} * NJ \quad (N \cong RF) \\ &\cong Rb \quad (\textit{strict presentation})\end{aligned}$$

But R is conservative, thence the counit $\bar{F}Rb \to b$ is an isomorphism; as a consequence R is fully faithful. Finally to see that F is dense, it is enough to consider the presentation $(\mathcal{K}, J, \varphi', \xi')$ with $\mathcal{K}$ and J as before, $\varphi'_b := \varphi_{Rb}$ and $\xi'_b := \xi_{Rb}$. This is a strict presentation since the original one was; thus F is dense by Proposition 2.6.

(2). Suppose first that R is an equivalence; note that in this case the left N-adjoint F can be defined simply as $R^{-1}N$ and $\bar{F}$ as R^{-1}. The only non trivial thing to prove is that $N \cong RF$. This is true iff $\bar{F}N \cong F$ (because $\bar{F} = R^{-1}$), but $\bar{F} \dashv R$ and $F \dashv_N R$ imply $\mathcal{B}(\bar{F}Na, b) \cong \mathcal{C}(Na, Rb) \cong \mathcal{B}(Fa, b)$ for each $b \in \mathcal{B}$ and thus $\bar{F}Na \cong Fa$.
Vice versa if R is conservative, preserves $\varphi_c * FJ$ and $N \cong RF$, by point (1) we only need to prove that $R\bar{F} \cong id_{\mathcal{C}}$. Let $c \in \mathcal{C}$; then $c \cong \varphi_c * NJ \cong \varphi_c * RFJ \cong R(\varphi_c * FJ) \cong R\bar{F}c$. $\square$

4. Beck's Theorem

In this section we are going to apply Theorem 3.5 in the context of monads to get a proof of Beck's theorem on monadic adjunctions.

Let (T, μ, η) be a monad in $\mathcal{V}$-Cat, where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\mu : T^2 \to T$ is the multiplication and $\eta : id_{\mathcal{C}} \to T$ the unit. Consider the Eilenberg-Moore category $\mathcal{C}^T$ of T-algebras whose objects are algebras $(c, \zeta : Tc \to c)$ in $\mathcal{C}$ and the hom-objects are defined as the equalizer

$$\mathcal{C}^T((c,\zeta_c),(d,\zeta_d)) \rightarrowtail \mathcal{C}(c,d) \overset{\mathcal{C}(\zeta_c, d)}{\underset{\mathcal{C}(Tc,\zeta_d)\circ T_{cd}}{\rightrightarrows}} \mathcal{C}(Tc,d)$$

Inside $\mathcal{C}^T$ we consider the Kleisli category $\mathcal{C}_T$ with objects free T-algebras, i.e. algebras of the form (Tc, μ_c) for some $c \in \mathcal{C}$.

Let $N : \mathcal{C}_T \to \mathcal{C}^T$ be the inclusion; we are going to show that N is dense by constructing a strict presentation. Before that, we first need to recall a general fact about algebras:

Proposition 4.1 ([ML71])**.** *For any T-algebra (c, ζ) in $\mathcal{C}^T$, $\zeta : Tc \to c$ is the split coequalizer in $\mathcal{C}$ of μ_c and $T\zeta$ with splitting maps $T^2c \xleftarrow{\eta_{Tc}} Tc \xleftarrow{\eta_c} c$. In particular ζ is an absolute coequalizer, i.e. it is preserved by each functor $G : \mathcal{C} \to \mathcal{B}$.*

Consider now the free $\mathcal{V}$-category $\mathcal{D}$ over the ordinary category

$$\{2 \overset{m}{\underset{s}{\rightrightarrows}} 1\};$$

and define a strict presentation for N as follows: $\mathcal{K} := \mathcal{D} \otimes \mathcal{C}^T$, the index functor $J : \mathcal{K} \to \mathcal{C}_T$ is such that:

$$J(i, -) := \mathcal{C}^T \xrightarrow{U^T} \mathcal{C} \xrightarrow{T^{i-1}} \mathcal{C} \xrightarrow{J} \mathcal{C}_T$$

where U^T is the forgetful functor, J is the canonical functor sending c to (Tc, μ_c) and $i = 1, 2$.
On the other side, given $\mathbf{c} = (c, \zeta)$ in $\mathcal{C}^T$, set

$$J(m, \mathbf{c}) := \mu_c : T^2\mathbf{c} \to T\mathbf{c};$$
$$J(s, \mathbf{c}) := T\zeta : T^2\mathbf{c} \to T\mathbf{c}.$$

The coefficient functor $\varphi : \mathcal{K}^{op} \otimes |\mathcal{C}^T| \to \mathcal{V}$ is defined as the composite

$$\mathcal{K}^{op} \otimes |\mathcal{C}^T| = \mathcal{D} \otimes \mathcal{C}^T \otimes |\mathcal{C}^T| \xrightarrow{\pi_{\mathcal{D}}} \mathcal{C}^T \otimes |\mathcal{C}^T| \xrightarrow{\mathcal{C}^T(-,|-|)} \mathcal{V}$$

Finally, let $\mathbf{c} = (c, \zeta)$ be an object of $\mathcal{C}^T$; then set $\xi_{\mathbf{c}}$ to be the isomorphism:

$$\begin{aligned}
\varphi_{\mathbf{c}} * NJ &\cong \operatorname*{colim}_{d \in \mathcal{D}} (\varphi(d, -, \mathbf{c}) * NJ(d, -)) \\
&\cong \operatorname*{colim}_{d \in \mathcal{D}} (\mathcal{C}^T(-, \mathbf{c}) * NJ(d, -)) \\
&\cong \operatorname*{colim}_{d \in \mathcal{D}} NJ(d, \mathbf{c}) \\
&\cong \operatorname{coeq}(\mu_c, T\zeta) \\
&\cong \mathbf{c} \qquad (\textit{by } 4.1)
\end{aligned}$$

We have just defined a presentation for N, to see that it is strict note that for each $\mathbf{a}$ in $\mathcal{C}_T$,

$$\varphi_{\mathbf{c}} * \widehat{J}\mathbf{a} \cong \varphi_{\mathbf{c}} * \mathcal{C}^T(N\mathbf{a}, NJ-) \cong \mathcal{C}^T(N\mathbf{a}, \varphi_{\mathbf{c}} * NJ)$$

since N is fully faithful and the coequalizer $\varphi_{\mathbf{c}} * NJ$ is absolute. This means exactly that $\bar{\xi}_{\mathbf{ca}}$ is also invertible for each $\mathbf{c} \in \mathcal{C}^T$ and $\mathbf{a} \in \mathcal{C}_T$; i.e. $(\mathcal{K}, J, \varphi, \xi)$ is a strict presentation.

An immediate consequence of this is that the inclusion $N : \mathcal{C}_T \to \mathcal{C}^T$ is dense; moreover we can apply Theorem 3.5 to get:

Theorem 4.2 (Beck)**.** *An adjunction $L \dashv R : \mathcal{B} \to \mathcal{C}$ is monadic iff R is conservative and creates coequalizers of R-split pairs (i.e. of those pairs (f, g) such that (Rf, Rg) has a split coequalizer in $\mathcal{C}$).*

Proof. One direction is done in the usual way and can be found for example in [ML71]. For the other direction, suppose that R is conservative and creates coequalizers of R-split pairs. Let $(T = RL, \mu = R\epsilon L, \eta)$ be the monad induced by the adjunction (η and ϵ are respectively the unit and counit); then the previous argument gives a strict presentation $(\mathcal{K}, J, \varphi, \xi)$ for the inclusion $N : \mathcal{C}_T \to \mathcal{C}^T$.
Let $K : \mathcal{B} \to \mathcal{C}^T$ be the comparison functor defined on objects by $Kb = (Rb, R\epsilon_b) \in \mathcal{C}^T$; to apply Theorem 3.5 we need to find, among the other things, an ordinary left adjoint

and a left N-adjoint to K. Let $U^T : \mathcal{C}^T \to \mathcal{C}$ be the forgetful functor; define then $F : \mathcal{C}^T \to \mathcal{B}$ as the coequalizer

$$\begin{tikzcd} LRLU^T \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\epsilon LU^T"'] & LU^T \arrow[r] & F \end{tikzcd}$$

where α is defined on components $\mathbf{c} = (c, \zeta) \in \mathcal{C}^T$ as $\alpha_{\mathbf{c}} := L\zeta : LRLc \to Lc$. Note that this coequalizer exists in $[\mathcal{C}^T, \mathcal{B}]$ since for each $(c, \zeta) \in \mathcal{C}^T$ the pair $(L\zeta, \epsilon_{Lc})$ is R-split (by 4.1) and R creates coequalizers of such pairs. Then $F \dashv K$; indeed for each $\mathbf{c} = (c, \zeta) \in \mathcal{C}^T$ and $b \in \mathcal{B}$ we have:

$$\begin{aligned} \mathcal{B}(F\mathbf{c}, b) &\cong \mathcal{B}(\mathrm{coeq}(L\zeta, \epsilon Lc), b) \\ &\cong \mathrm{eq}(\mathcal{B}(L\zeta, b), \mathcal{B}(\epsilon Lc, b)) \\ &\cong \mathrm{eq}(\mathcal{C}(\zeta, Rb), \mathcal{C}(RLc, R\epsilon b) \circ RL) \quad (L \dashv R) \\ &= \mathcal{C}^T(\mathbf{c}, Kb); \end{aligned}$$

As a consequence $FN \dashv_N R$ and, since $KF \cong id_{\mathcal{C}^T}$ by construction, $N \cong K(FN)$. Moreover K is conservative and preserves the colimits $\varphi_c * FNJ \cong \mathrm{coeq}(F\mu_c, FT\zeta)$ for each $\mathbf{c} = (c, \zeta) \in \mathcal{C}^T$ since R does; hence we can apply Theorem 3.5(2) and get that K is an equivalence. $\square$

5. Free Cocompletions

Here we apply the theory of density presentations to get information on the free cocompletions of categories under a certain class of colimits. The results we show are taken from [Kel82], but a more detailed treatment of such cocompletions and their properties can be found in [KS05].

Given a category $\mathcal{A}$, it's reasonable to look for its free cocompletion (under a class of colimits) in the category of presheaves $[\mathcal{A}^{op}, \mathcal{V}]$ like it happens in the ordinary case. The problem is that we want the free cocompletion to be a $\mathcal{V}$-category like $\mathcal{A}$, but $[\mathcal{A}^{op}, \mathcal{V}]$ may not be (as we said in section 1). To avoid this, instead of considering all presheaves we can restrict ourselves to the small ones:

Definition 5.1. Let $\mathcal{A}$ be a category; a presheaf $F : \mathcal{A}^{op} \to \mathcal{V}$ is called small if there exists a small $\mathcal{K}$ and a functor $J : \mathcal{K} \to \mathcal{A}$ such that $F \cong \mathrm{Lan}_{J^{op}} H \cong H * \mathcal{A}(-, J)$ for some $H : \mathcal{K}^{op} \to \mathcal{V}$. Denote by $\mathscr{P}\mathcal{A}$ the full subcategory of $[\mathcal{A}^{op}, \mathcal{V}]$ whose objects are small functors.

It is interesting to note (Proposition 4.83 in [Kel82]) that if $F : \mathcal{A} \to \mathcal{V}$ is small the functor J in the definition can be chosen to be fully faithful, so that F is the left Kan extension of its restriction to a small full subcategory of $\mathcal{A}$.

As we anticipated, the category $\mathscr{P}\mathcal{A}$ of small presheaves is a $\mathcal{V}$-category: for each F small and any G (using the universal property of Lan) we have

$$[\mathcal{A}^{op}, \mathcal{V}](F, G) \cong [\mathcal{A}^{op}, \mathcal{V}](\mathrm{Lan}_{J^{op}} H, G) \cong [\mathcal{K}^{op}, \mathcal{V}](H, GJ^{op}),$$

which exists in $\mathcal{V}$ since $\mathcal{K}$ is small. Moreover the Yoneda embedding $Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathcal{V}]$ factors through $\mathscr{P}\mathcal{A}$ since for each $a \in \mathcal{A}$ we have $\mathcal{A}(-, a) \cong H * \mathcal{A}(-, J_a)$, where $H = I : \mathcal{I}^{op} \to \mathcal{V}$ and $J_a : \mathcal{I} \to \mathcal{A}$ points to a ($\mathcal{I}$ being the unit $\mathcal{V}$-category). As a consequence we can see $\mathcal{A}$ as a full subcategory of $\mathscr{P}\mathcal{A}$, which is moreover cocomplete:

Proposition 5.2 ([Kel82])**.** *$\mathscr{P}\mathcal{A}$ is closed in $[\mathcal{A}^{op}, \mathcal{V}]$ under small colimits.*

Let us now fix a collection Φ of functors with small domains; a colimit with a weight from Φ will be called Φ-colimit. Our aim is to construct the free cocompletion of a category $\mathcal{A}$ under Φ-colimits.

Definition 5.3. We say that a category $\mathcal{C}$ is Φ-cocomplete if it has all Φ-colimits and that a functor $F : \mathcal{C} \to \mathcal{B}$ is Φ-cocontinuous if preserves them.

Given a category $\mathcal{A}$, consider the closure $\Phi(\mathcal{A})$ of $\mathcal{A}$ in $\mathscr{P}\mathcal{A}$ under Φ-colimits and denote by $N : \mathcal{A} \to \Phi(\mathcal{A})$ the inclusion; then the following result holds.

Theorem 5.4. *$\Phi(\mathcal{A})$ is the free Φ-cocompletion of $\mathcal{A}$; i.e. $\Phi(\mathcal{A})$ is Φ-cocomplete and for each Φ-cocomplete $\mathcal{C}$ the left Kan extension along N induces an equivalence*

$$\mathrm{Lan}_N : [\mathcal{A}, \mathcal{C}] \longrightarrow \Phi\text{-}\mathrm{Coct}[\Phi(\mathcal{A}), \mathcal{B}]$$

with inverse $[N, 1]$. Moreover the inclusion $N : \mathcal{A} \to \Phi(A)$ is dense, with discrete presentation given by the family of all Φ-colimits.

Proof. First we prove that N is dense. To see that the family of all Φ-colimits is a discrete presentation for N we only need to prove that they are N-absolute (since $\Phi(\mathcal{A})$ is by definition the closure of $\mathcal{A}$ under them). But $\widetilde{N} : \Phi(\mathcal{A}) \to [\mathcal{A}^{op}, \mathcal{V}]$ is just the inclusion and hence preserves Φ-colimits by construction. As a consequence N is dense by Theorem 2.8. Now, $\Phi(\mathcal{A})$ is Φ-cocomplete by definition and Lan_N is fully faithful since:

$$[\Phi(\mathcal{A}), \mathcal{B}](\mathrm{Lan}_N F, \mathrm{Lan}_N G) \cong [\mathcal{A}, \mathcal{B}](F, (\mathrm{Lan}_N G)N) \cong [\mathcal{A}, \mathcal{B}](F, G)$$

where the first isomorphism is given by the universal property of Lan and the latter follows from $(\mathrm{Lan}_N G)N \cong G$ (N is fully faithful). Finally, by Corollary 3.2, for each Φ-cocontinuous $S : \Phi(\mathcal{A}) \to \mathcal{B}$ the identity id_{SN} exhibits S as $\mathrm{Lan}_N(SN)$; this shows that $[N, 1]$ is a left-inverse for Lan_N which is thence an equivalence of categories. $\square$

6. Absolutely Dense Functors

Let $N : \mathcal{A} \to \mathcal{C}$ be any dense functor; then Theorem 2.8 gives a canonical presentation for N and shows that the colimits involved $\widetilde{N}c * N$ are N-absolute. In this section we consider those dense functors for which $\widetilde{N}c * N$ are furthermore preserved by any functor.

Definition 6.1. A functor $N : \mathcal{A} \to \mathcal{C}$ is called **absolutely dense** if it is dense and all colimits $\widetilde{N}c * N$, $c \in \mathcal{C}$, are absolute, i.e. are preserved by any functor.

We can classify absolutely dense functors as follows:

Proposition 6.2. *Given $N : \mathcal{A} \to \mathcal{C}$, the following are equivalent:*

(1) *N is absolutely dense;*
(2) *for any category $\mathcal{B}$*
$$[N, 1] : [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$$
is fully faithful;
(3) $[N^{op}, 1] : [\mathcal{C}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$ *is fully faithful;*
(4) *for every $\varphi : \mathcal{C}^{op} \to \mathcal{V}$ and $F : \mathcal{C} \to \mathcal{B}$ the colimit $\varphi * F$ exists iff $(\varphi N^{op}) * (FN)$ exists and they are isomorphic.*

Proof. (1) $\Rightarrow$ (2). Let $F, G : \mathcal{C} \to \mathcal{B}$ be any pair of functors; since N is absolutely dense, by Corollary 3.2, $F \cong \mathrm{Lan}_N(FN)$ and hence:
$$\begin{aligned}[\mathcal{C}, \mathcal{B}](F, G) &\cong [\mathcal{C}, \mathcal{B}](\mathrm{Lan}_N(FN), G)\\ &\cong [\mathcal{A}, \mathcal{B}](FN, GN)\end{aligned}$$
by the universal property of Lan. Then $[N, 1]$ is fully faithful.

(2) $\Rightarrow$ (3). Take $\mathcal{B} = \mathcal{V}^{op}$ and then use $[N^{op}, 1] = [N, 1]^{op}$.

(3) $\Rightarrow$ (1). N is dense since $\widetilde{N}$ is the composite $[N^{op}, 1]Y$ and $[N^{op}, 1]$ is full and faithful by hypothesis. Moreover, for a fixed c in $\mathcal{C}$, any functor $F : \mathcal{C} \to \mathcal{B}$, and $b \in \mathcal{B}$ we have:
$$\begin{aligned}\mathcal{B}(\widetilde{N}c * (FN), b) &\cong [\mathcal{A}^{op}, \mathcal{V}](\widetilde{N}c, \mathcal{B}(FN-, b))\\ &\cong [\mathcal{A}^{op}, \mathcal{V}](\mathcal{C}(-, c)N^{op}, \mathcal{B}(F-, b)N^{op})\\ &\cong [\mathcal{C}^{op}, \mathcal{V}](\mathcal{C}(-, c), \mathcal{B}(F-, b))\\ &\cong \mathcal{B}(Fc, b)\\ &\cong \mathcal{B}(F(\widetilde{N}c * N), b)\end{aligned}$$
so that the colimits $\widetilde{N}c * N$ are absolute and hence N is absolutely dense.

(3) $\Rightarrow$ (4). For any $b \in \mathcal{B}$ we have:

$$\begin{aligned}\mathcal{B}((\varphi N^{op}) * (FN), b) &\cong [\mathcal{A}^{op}, \mathcal{V}](\varphi N^{op}, \mathcal{B}(FN-, b))\\ &\cong [\mathcal{A}^{op}, \mathcal{V}](\varphi N^{op}, \mathcal{B}(F-, b)N^{op})\\ &\cong [\mathcal{C}^{op}, \mathcal{V}](\varphi, \mathcal{B}(F-, b))\\ &\cong \mathcal{B}(\varphi * F, b).\end{aligned}$$

Then (4) follows.

(4) $\Rightarrow$ (3). Taking as F the Yoneda embedding $Y : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$, (4) says that for each $\varphi : \mathcal{C}^{op} \to \mathcal{V}$ we have $\varphi \cong \mathrm{Lan}_{N^{op}}(\varphi N^{op})$. Thence we can conclude as in (1) $\Rightarrow$ (2). □

Functors satisfying property (3) of the last Proposition are called connected in [BV02] and are used to classify reflective and coreflective subcategories of presheaves:

Theorem 6.3 (3.11 in [BV02])**.** *Let $\mathcal{A}$ be a category and $J : \mathcal{K} \to [\mathcal{A}^{op}, \mathcal{V}]$ a full embedding of a reflective and coreflective subcategory (i.e. such that the inclusion J has both a left and a right adjoint). Then there exists $N : \mathcal{A} \to \mathcal{C}$ connected such that J is isomorphic to the embedding*

$$[N^{op}, 1] : [\mathcal{C}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}].$$

Proof. (Sketch). We are only going to show how to define $N : \mathcal{A} \to \mathcal{C}$. Let $L : [\mathcal{A}^{op}, \mathcal{V}] \to \mathcal{K}$ be the left adjoint of J and consider the functor E given by the composite

$$\mathcal{A} \xrightarrow{Y} [\mathcal{A}^{op}, \mathcal{V}] \xrightarrow{L} \mathcal{K},$$

then we can factorise it as $E = E'N$ with $N : \mathcal{A} \to \mathcal{B}$ surjective on objects and $E' : \mathcal{B} \to \mathcal{K}$ fully faithful. One can prove that N is absolutely dense, hence connected, and $J \cong [N^{op}, 1]$. □

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