MUNI

Dualities in the theory of Accessible Categories

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Athens, 2 April 2023

PSSL 107

Gabriel–Ulmer duality

$$\mathrm{Lfp}(-, \mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set})$$

Gabriel–Ulmer duality

$$\mathrm{Lfp}(-, \mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set})$$

Lex:

- $\mathcal{C}$ small with finite limits;
- $F : \mathcal{C} \to \mathcal{C}'$ preserving
  - no specified colimits;
  - finite limits.
- natural transformations.

Gabriel–Ulmer duality

$$\mathrm{Lfp}(-,\mathbf{Set})\colon \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-,\mathbf{Set})$$

Lfp:

- $\mathcal{K}$ complete + freely generated by a small category under filtered colimits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - limits;
  - filtered colimits.
- natural transformations.

Lex:

- $\mathcal{C}$ small with finite limits;
- $F : \mathcal{C} \to \mathcal{C}'$ preserving
  - no specified colimits;
  - finite limits.
- natural transformations.

Diers duality

$$\text{Lmfp}(-, \mathbf{Set}) : \mathbf{Lmfp} \rightleftarrows \mathbf{Fam\text{-}Ex}^{op} : \text{Fam-Ex}(-, \mathbf{Set})$$

Lmfp:

- $\mathcal{K}$ finitely accessible with connected limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - connected limits
  - filtered colimits;
- natural transformations.

Diers duality

$$\mathrm{Lmfp}(-,\mathbf{Set})\colon \mathbf{Lmfp} \rightleftarrows \mathbf{Fam\text{-}Ex}^{op} : \mathrm{Fam\text{-}Ex}(-,\mathbf{Set})$$

Lmfp:

- $\mathcal{K}$ finitely accessible with connected limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - connected limits
  - filtered colimits;
- natural transformations.

Fam-Lex:

- $\mathcal{E} \simeq \mathrm{Fam}(\mathcal{C})$ with finite limits, $\mathcal{C}$ small;
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - coproducts
  - finite limits;
- natural transformations.

Examples

Ψ-limits commuting with $\mathfrak{C}$-colimits in **Set**

1. $\Psi = \{\text{small categories}\}$ $\rightarrow$ $\mathfrak{C} = \emptyset$;
2. $\Psi = \{\text{connected diagrams}\}$ $\rightarrow$ $\mathfrak{C} = \{\text{discrete categories}\}$;

Examples

Ψ-limits commuting with $\mathfrak{C}$-colimits in **Set**

1. $\Psi = \{\text{small categories}\}$ $\rightarrow$ $\mathfrak{C} = \emptyset$;
2. $\Psi = \{\text{connected diagrams}\}$ $\rightarrow$ $\mathfrak{C} = \{\text{discrete categories}\}$;
3. $\Psi = \emptyset$ $\rightarrow$ $\mathfrak{C} = \{\text{small categories}\}$;
4. $\Psi = \{\text{finite categories}\}$ $\rightarrow$ $\mathfrak{C} = \{\text{filtered categories}\}$;
5. $\Psi = \{\text{finite discrete categories}\}$ $\rightarrow$ $\mathfrak{C} = \{\text{sifted categories}\}$;
6. $\Psi = \{\emptyset\}$ $\rightarrow$ $\mathfrak{C} = \{\text{connected categories}\}$;
7. $\Psi = \{\text{discrete categories}\}$ $\rightarrow$ $\mathfrak{C} = ??$
8. $\Psi = \{\text{wide pullbacks}\}$ $\rightarrow$ $\mathfrak{C} = ??$

Companions

A colimit type $\mathfrak{C}$ is the data of a full subcategory $\mathfrak{C}_{\mathcal{D}} \subseteq [\mathcal{C}, \mathbf{Set}]$ for each small category $\mathcal{D}$.

Definition (for today)

Let Ψ be a class of indexing categories; a colimit type $\mathfrak{C}$ is a companion for Ψ if:

- Ψ-limits commute with $\mathfrak{C}$-colimits in **Set**: for each $\mathcal{D}$, any $\mathcal{B} \in \Psi$ and

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\forall H} & [\mathcal{D}, \mathbf{Set}] \\ & \searrow \quad \nearrow & \\ & \mathfrak{C}_{\mathcal{D}} & \end{array} \qquad \operatorname{colim}_{c \in \mathcal{D}} \ \lim_{b \in \mathcal{B}} \ H(b, c) \cong \lim_{b \in \mathcal{B}} \ \operatorname{colim}_{c \in \mathcal{D}} \ H(b, c)$$

- Let $\mathcal{K}$ be finitely accessible with Ψ-limits and $F \colon \mathcal{K} \to \mathbf{Set}$ preserve Ψ-limits and filtered colimits. Then

$$F \cong \operatorname{colim}_i \mathcal{K}(X_i, -)$$

  is a $\mathfrak{C}$-colimit of representables with $X_i \in \mathcal{K}_f$.

Back to the examples

7. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} := \{\text{pseudo equivalence relations}\}$

8. $\Psi = \{\text{wide pullbacks}\}$ - $\mathfrak{F} := \{\text{free groupoid actions}\}$

Back to the examples

7. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} :=$ {pseudo equivalence relations}

   $\mathfrak{R}_{\mathcal{D}} \neq \emptyset$ iff $\mathcal{D} = \{\rightrightarrows\}$; in that case $\mathfrak{R}_{\mathcal{D}} \subseteq \mathbf{Set}^{\rightrightarrows}$ consists of pairs

   $$X \xrightarrow{e} Z \underset{k}{\overset{h}{\rightrightarrows}} Y$$

   where e is epi and (h, k) a kernel pair.

8. $\Psi = \{\text{wide pullbacks}\}$ - $\mathfrak{F} := \{\text{free groupoid actions}\}$

Back to the examples

7. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} := \{\text{pseudo equivalence relations}\}$

   $\mathfrak{R}_{\mathcal{D}} \neq \emptyset$ iff $\mathcal{D} = \{\rightrightarrows\}$; in that case $\mathfrak{R}_{\mathcal{D}} \subseteq \mathbf{Set}^{\rightrightarrows}$ consists of pairs

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   where e is epi and (h, k) a kernel pair.

8. $\Psi = \{\text{wide pullbacks}\}$ - $\mathfrak{F} := \{\text{free groupoid actions}\}$

   $\mathfrak{F}_{\mathcal{D}} \neq \emptyset$ iff $\mathcal{D}$ is a groupoid, in that case $H \colon \mathcal{D} \to \mathbf{Set}$ is in $\mathfrak{F}_{\mathcal{D}}$ iff:

$$0 \longrightarrow HA \underset{Hg}{\overset{Hf}{\rightrightarrows}} HB$$

   is an equalizer for any $f, g \colon A \to B$ with $f \neq g$.

The Duality

Theorem

Let $\mathfrak{C}$ be a companion for Ψ. The following is a biequivalence of 2-categories

$$\mathrm{fAcc}_{\Psi}(-,\mathbf{Set}) \colon \mathbf{fAcc}_{\Psi} \rightleftarrows \mathfrak{C}\text{-}\mathbf{Ex}^{op} \colon \mathfrak{C}\text{-}\mathrm{Ex}(-,\mathbf{Set})$$

The Duality

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$$\mathrm{fAcc}_\Psi(-,\mathbf{Set})\colon \mathbf{fAcc}_\Psi \rightleftarrows \mathfrak{C}\text{-}\mathbf{Ex}^{op} : \mathfrak{C}\text{-}\mathrm{Ex}(-,\mathbf{Set})$$

$\mathbf{fAcc}_\Psi$:

- $\mathcal{K}$ finitely accessible with Ψ-limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - Ψ-limits
  - filtered colimits;
- natural transformations.

The Duality

Theorem

Let $\mathfrak{C}$ be a companion for Ψ. The following is a biequivalence of 2-categories

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$\mathbf{fAcc}_\Psi$:

- $\mathcal{K}$ finitely accessible with Ψ-limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - Ψ-limits
  - filtered colimits;
- natural transformations.

$\mathfrak{C}$-Ex:

- $\mathcal{E} \simeq \mathfrak{C}(\mathcal{C})$ with finite limits, $\mathcal{C}$ small;
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - $\mathfrak{C}$-colimits
  - finite limits;
- natural transformations.

The Duality

Theorem (Enriched)

Let $\mathfrak{C}$ be a companion for Ψ. The following is a biequivalence of 2-categories

$$\mathrm{fAcc}_\Psi(-,\mathcal{V})\colon \mathbf{fAcc}_\Psi \rightleftarrows \mathfrak{C}\text{-}\mathbf{Ex}^{op} : \mathfrak{C}\text{-}\mathrm{Ex}(-,\mathcal{V})$$

$\mathbf{fAcc}_\Psi$:

- $\mathcal{K}$ finitely accessible $\mathcal{V}$-category with Ψ-limits;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - Ψ-limits
  - flat colimits;
- $\mathcal{V}$-natural transformations.

$\mathfrak{C}$-$\mathbf{Ex}$:

- $\mathcal{E} \simeq \mathfrak{C}(\mathcal{C})$ with finite limits, $\mathcal{C}$ small $\mathcal{V}$-category;
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - $\mathfrak{C}$-colimits
  - finite limits;
- natural transformations.

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\rightarrow$ Gabriel–Ulmer

$$\mathrm{Lfp}(-, \mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set})$$

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\rightarrow$ Gabriel–Ulmer
2. $\Psi = \{\text{connected diagrams}\}$ - $\mathfrak{C} = \{\text{discrete categories}\}$ $\rightarrow$ Diers

$$\mathrm{Lmfp}(-, \mathbf{Set}) : \mathbf{Lmfp} \rightleftarrows \mathbf{Fam\text{-}Ex}^{op} : \mathrm{Fam\text{-}Ex}(-, \mathbf{Set})$$

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\rightarrow$ Gabriel–Ulmer
2. $\Psi = \{\text{connected diagrams}\}$ - $\mathfrak{C} = \{\text{discrete categories}\}$ $\rightarrow$ Diers
3. $\Psi = \emptyset$ - $\mathfrak{C} = \{\text{small categories}\}$ $\rightarrow$ Makkai–Paré

$$\mathrm{fAcc}(-,\mathbf{Set})\colon \mathbf{fAcc} \rightleftarrows \mathcal{P}\text{-}\mathbf{Ex}^{op} : \mathcal{P}\text{-}\mathrm{Ex}(-,\mathbf{Set})$$

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\longrightarrow$ Gabriel–Ulmer
2. $\Psi = \{\text{connected diagrams}\}$ - $\mathfrak{C} = \{\text{discrete categories}\}$ $\longrightarrow$ Diers
3. $\Psi = \emptyset$ - $\mathfrak{C} = \{\text{small categories}\}$ $\longrightarrow$ Makkai–Paré
4. $\Psi = \{\text{finite categories}\}$ - $\mathfrak{C} = \{\text{filtered categories}\}$ $\longrightarrow$ *New*

$$\mathrm{fAcc}_\Psi(-,\mathbf{Set})\colon \mathbf{fAcc}_\Psi \rightleftarrows \mathbf{fAcc}_\Psi^{op} \colon \mathrm{fAcc}_\Psi(-,\mathbf{Set})$$

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\longrightarrow$ Gabriel–Ulmer
2. $\Psi = \{\text{connected diagrams}\}$ - $\mathfrak{C} = \{\text{discrete categories}\}$ $\longrightarrow$ Diers
3. $\Psi = \emptyset$ - $\mathfrak{C} = \{\text{small categories}\}$ $\longrightarrow$ Makkai–Paré
4. $\Psi = \{\text{finite categories}\}$ - $\mathfrak{C} = \{\text{filtered categories}\}$ $\longrightarrow$ *New*
5. $\Psi =$ small sound class - $\mathfrak{C} = \Psi$-filtered cats. $\longrightarrow$ *New* (enriched)

$$\mathrm{fAcc}_\Psi(-,\mathcal{V})\colon \mathbf{fAcc}_\Psi \rightleftarrows \Psi\text{-}\mathbf{Acc}_{lex}^{op} : \Psi\text{-}\mathrm{Acc}_{lex}(-,\mathcal{V})$$

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\to$ Gabriel–Ulmer
2. $\Psi = \{\text{connected diagrams}\}$ - $\mathfrak{C} = \{\text{discrete categories}\}$ $\to$ Diers
3. $\Psi = \emptyset$ - $\mathfrak{C} = \{\text{small categories}\}$ $\to$ Makkai–Paré
4. $\Psi = \{\text{finite categories}\}$ - $\mathfrak{C} = \{\text{filtered categories}\}$ $\to$ *New*
5. $\Psi =$ small sound class - $\mathfrak{C} = \Psi$-filtered cats. $\to$ *New* (enriched)
6. $\Psi =$ small sound class - $\mathfrak{C} = \Psi$-filtered cats. $\to$ *New* (infinitary)

$$\alpha\text{-Acc}_\Psi(-,\mathbf{Set})\colon \alpha\text{-}\mathbf{Acc}_\Psi \rightleftarrows \Psi\text{-}\mathbf{Acc}_\alpha^{op} : \Psi\text{-Acc}_\alpha(-,\mathbf{Set})$$

Examples

1. $\Psi = \{\text{small categories}\}$ - $\mathfrak{C} = \emptyset$ $\longrightarrow$ Gabriel–Ulmer
2. $\Psi = \{\text{connected diagrams}\}$ - $\mathfrak{C} = \{\text{discrete categories}\}$ $\longrightarrow$ Diers
3. $\Psi = \emptyset$ - $\mathfrak{C} = \{\text{small categories}\}$ $\longrightarrow$ Makkai–Paré
4. $\Psi = \{\text{finite categories}\}$ - $\mathfrak{C} = \{\text{filtered categories}\}$ $\longrightarrow$ *New*
5. $\Psi =$ small sound class - $\mathfrak{C} = \Psi$-filtered cats. $\longrightarrow$ *New* (enriched)
6. $\Psi =$ small sound class - $\mathfrak{C} = \Psi$-filtered cats. $\longrightarrow$ $(\alpha = \infty)$

$$\mathrm{CCat}_{\Psi}(-, \mathbf{Set}) \colon \mathbf{CCat}_{\Psi} \rightleftarrows \Psi\text{-}\mathbf{Lp}^{op} \colon \Psi\text{-}\mathrm{Lp}(-, \mathbf{Set})$$

Examples (continued)

7. $\Psi = \{\text{wide pulbacks}\}$ - $\mathfrak{F} = \{\text{free gpd. act.}\}$ $\rightarrow$ Hu–Tholen

$$\mathrm{Lfpp}(-,\mathbf{Set})\colon \mathbf{Lfpp} \rightleftarrows \mathfrak{F}\text{-}\mathbf{Ex}^{op} : \mathfrak{F}\text{-}\mathrm{Ex}(-,\mathbf{Set})$$

Examples (continued)

7. $\Psi = \{\text{wide pulbacks}\}$ - $\mathfrak{F} = \{\text{free gpd. act.}\}$ $\rightarrow$ Hu–Tholen
8. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} = \{\text{ps. equiv. rel.}\}$ $\rightarrow$ Hu

$$\mathrm{wLfp}(-, \mathbf{Set}) : \mathbf{wLfp} \rightleftarrows \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-, \mathbf{Set})$$

wLfp:

- $\mathcal{K}$ finitely accessible with products;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - products
  - filtered colimits;
- natural transformations.

Examples (continued)

7. $\Psi = \{\text{wide pulbacks}\}$ - $\mathfrak{F} = \{\text{free gpd. act.}\}$ $\rightarrow$ Hu–Tholen
8. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} = \{\text{ps. equiv. rel.}\}$ $\rightarrow$ Hu

$$\mathrm{wLfp}(-,\mathbf{Set})\colon \mathbf{wLfp} \rightleftarrows \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-,\mathbf{Set})$$

wLfp:

- $\mathcal{K}$ finitely accessible with products;
- $F : \mathcal{K} \to \mathcal{K}'$ preserving
  - products
  - filtered colimits;
- natural transformations.

p-Ex:

- $\mathcal{E}$ small Barr-exact with enough projectives;
- $F : \mathcal{E} \to \mathcal{E}'$ preserving
  - regular epimorphisms
  - finite limits;
- natural transformations.

Examples (continued)

7. $\Psi = \{$wide pulbacks$\}$ - $\mathfrak{F} = \{$free gpd. act.$\}$ $\rightarrow$ Hu–Tholen
8. $\Psi = \{$discrete categories$\}$ - $\mathfrak{R} = \{$ps. equiv. rel.$\}$ $\rightarrow$ Hu
9. $\Psi = \{$discrete categories & proj. powers$\}$ - $\mathfrak{R} = \{$ps. equiv. rel.$\}$ $\rightarrow$ *New* (enriched)

$$\mathrm{wLfp}(-,\mathcal{V}) : \mathbf{wLfp} \rightleftarrows \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-,\mathcal{V})$$

Examples (continued)

7. $\Psi = \{\text{wide pulbacks}\}$ - $\mathfrak{F} = \{\text{free gpd. act.}\}$ $\rightarrow$ Hu–Tholen
8. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} = \{\text{ps. equiv. rel.}\}$ $\rightarrow$ Hu
9. $\Psi = \{\text{discrete categories \& proj. powers}\}$ - $\mathfrak{R} = \{\text{ps. equiv. rel.}\}$ $\rightarrow$ *New* (enriched)

$$\begin{array}{ccc}
\mathrm{wLfp}(-,\mathcal{V}) : \mathbf{wLfp} & \rightleftarrows & \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-,\mathcal{V}) \\
 & & \big\downarrow \\
 & & \mathbf{Ex}^{op} : \mathrm{Reg}(-,\mathcal{V})
\end{array}$$

Examples (continued)

7. $\Psi = \{\text{wide pulbacks}\}$ - $\mathfrak{F} = \{\text{free gpd. act.}\}$ $\rightarrow$ Hu–Tholen
8. $\Psi = \{\text{discrete categories}\}$ - $\mathfrak{R} = \{\text{ps. equiv. rel.}\}$ $\rightarrow$ Hu
9. $\Psi = \{\text{discrete categories } \& \text{ proj. powers}\}$ - $\mathfrak{R} = \{\text{ps. equiv. rel.}\}$ $\rightarrow$ *New* (enriched)

$$
\begin{array}{ccc}
\mathrm{wLfp}(-,\mathcal{V})\colon \mathbf{wLfp} & \rightleftarrows & \mathbf{p\text{-}Ex}^{op} : \mathrm{Reg}(-,\mathcal{V}) \\
\downarrow & & \downarrow \\
\mathrm{Def}(-,\mathcal{V})\colon \mathbf{Def} & \rightleftarrows & \mathbf{Ex}^{op} : \mathrm{Reg}(-,\mathcal{V})
\end{array}
$$

Thank You!