Equivalent Characterizations of Accessible V-categories

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Outline:
1) Accessible Categories
2) V-categories
3) Accessible V-categories: 2 notions
4) The Characterization Theorem

1) Accessible categories

Def: Let α be a regular cardinal; a category $\mathcal{A}$ is called α-accessible if it is the free completion of a small category under α-filtered colimits. Accessible = α-accessible for some α.

→ $\mathcal{C}$ is α-filtered if every α-small diagram has a cocone.

→ Define $\mathcal{A}_\alpha$: $X \in \mathcal{A}$ s.t. $\mathcal{A}(X,-)$ preserves α-filtered colimits.

$\mathcal{A}$ is α-accessible $\iff$ 1) $\mathcal{A}$ has α-filt. colimits; 2) $\mathcal{A}_\alpha$ is small and every $A \in \mathcal{A}$ is an α-filt. colimit of objects from $\mathcal{A}_\alpha$.

Examples: 1) Set, Ab, R-Mod, ... [illegible] locally presentable
2) Cat, Grp, ...
3) [illegible]
4) [illegible]
5) [illegible]
6) Models of first order theories

Important: 1) [illegible]
2) [illegible]

2) V-categories

$\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is symmetric monoidal closed, complete and cocomplete.

Ordinary Category	V-category
A category $\mathcal{K}$ is the data of:	A V-category $\mathcal{K}$ is the data of:
1) a set of objects $Ob(\mathcal{K})$	1) a set of objects $Ob(\mathcal{K})$
2) $\forall X, Y$ a set $\mathcal{K}(X,Y)$ of morphisms	2) $\forall X, Y$ an object $\mathcal{K}(X,Y) \in \mathcal{V}$
3) identities	3) identities
4) composition: $\mathcal{K}(Y,Z) \times \mathcal{K}(X,Y) \to \mathcal{K}(X,Z)$	4) composition: $\mathcal{K}(Y,Z) \otimes \mathcal{K}(X,Y) \to \mathcal{K}(X,Z)$
... axioms	... axioms

V-functors: A V-functor $F: \mathcal{A} \to \mathcal{K}$ between V-categories is the data of: 1) $F: Ob(\mathcal{A}) \to Ob(\mathcal{K})$; 2) $F_{XY}: \mathcal{A}(X,Y) \to \mathcal{K}(FX,FY)$ in $\mathcal{V}$ for any $X, Y \in \mathcal{A}$.

→ V-natural transformations
→ $[\mathcal{C}, \mathcal{A}]$: V-category of V-functors
→ $Y: \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$ Yoneda

Accessible (locally presentable)

Examples: 1) $(Set, \times)$ ← ordinary categories
2) $(Ab, \otimes)$ ← preadditive categories
3) $(Cat, \times)$ ← 2-categories
4) $(sSet, \times)$ ← simplicial categories
5) $(DGAb, \otimes)$ ← dg-categories
6) $(2, \wedge)$
7) $([0,\infty], +)$

Note: Every V-category $\mathcal{A}$ has an associated ordinary category $\mathcal{A}_0$: $Ob(\mathcal{A}_0) = Ob(\mathcal{A})$, $\mathcal{A}_0(A,B) = \mathcal{V}_0(I, \mathcal{A}(A,B))$.

Weighted limits:
→ Ordinary colimits are not enough, we need weights $\varphi: \mathcal{C}^{op} \to \mathcal{V}$

Def: Given $\varphi: \mathcal{C}^{op} \to \mathcal{V}$, $H: \mathcal{C} \to \mathcal{A}$, the colimit of H weighted by φ is an object $colim_\varphi H \in \mathcal{A}$ for which

$$\mathcal{A}(colim_\varphi H, A) \cong [\mathcal{C}^{op}, \mathcal{V}](\varphi -, \mathcal{A}(H-,A))$$
cylinders

V-naturally in $A \in \mathcal{A}$.

Conical colimits: If $\mathcal{C}$ is an ordinary category, we can consider the weight $\Delta I: \mathcal{C}_\mathcal{V}^{op} \to \mathcal{V}$. Thus for any $H: \mathcal{C}_\mathcal{V} \to \mathcal{A}$ (i.e. $\bar{H}: \mathcal{C} \to \mathcal{A}_0$) we define $colim\, H := colim_{\Delta I}(H) \in \mathcal{A}$

$$\mathcal{A}(colim\, H, A) \cong \lim \mathcal{A}(H-, A)$$
in $\mathcal{V}_0$

Copowers (tensors): Now $\mathcal{C} = \mathcal{I} = \{*\}$, $\mathcal{I}(*,*) = I$. Then $\varphi: \mathcal{I}^{op} \xrightarrow{X} \mathcal{V}$ and $H: \mathcal{I} \xrightarrow{A} \mathcal{A}$; then we define $X \cdot A := colim_\varphi H$

$$\mathcal{A}(X \cdot A, B) \cong [X, \mathcal{A}(A,B)] \quad \text{in } \mathcal{V}_0$$

Similarly one defines weighted limits, conical limits, and powers.

3) Accessible V-categories

→ $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ sym. mon. closed complete, cocomplete + accessible

Take α_0 s.t. $(\mathcal{V}_0)_{\alpha_0}$ contains I and is closed under $\otimes$
From now on $\alpha \geq \boxed{\alpha_0}$

Def: We say that a V-category $\mathcal{A}$ is α-accessible if it is the free cocompletion of a small V-category under < ______ >
$\mathcal{A}$ is accessible if it is α-accessible for some α

I) < conical α-filtered colimits >
i.e. colimits indexed by α-filtered ordinary categories
Call this conically α-accessible ← good, bad

?$\|$?

II) < colimits weighted by α-flat V-functors > bad, good
→ $\varphi: \mathcal{C}^{op} \to \mathcal{V}$ is α-flat if $colim_\varphi(-): [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves α-small limits in $\mathcal{V}$ → { α-small conical limits + powers by $X \in (\mathcal{V}_0)_\alpha$ }
Call these: α-accessible ← [Borceux, Quinteiro, Rosický]

$$\{\text{conical } \alpha\text{-filtered colimits}\} \subsetneq \left\{\begin{matrix}\text{colimits weighted by} \\ \alpha\text{-flat V-functors}\end{matrix}\right\}$$

In general:

$$\begin{tikzcd}
 & \mathcal{A}\ \alpha\text{-accessible} \arrow[dl, bend right, "?"'] \arrow[dr, bend left, "?"] & \\
\mathcal{A} \text{ conically } \alpha\text{-accessible} \arrow[rr, Rightarrow] & & \mathcal{A}_0\ \alpha\text{-accessible}
\end{tikzcd}$$

Forgetting α:

$$\begin{tikzcd}
 & \mathcal{A} \text{ accessible} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow, "{[BQR]}"] & \\
\mathcal{A} \text{ conically accessible} \arrow[rr, Rightarrow] & & \boxed{\mathcal{A}_0 \text{ accessible}}
\end{tikzcd}$$

Prop: [BQR] Let $\mathcal{V} = CMon, Ab, R\text{-}Mod$. Then

$\mathcal{A}$ is α-accessible $\iff$ { $\mathcal{A}$ is conically α-accessible and has finite direct sums }

Prop: Let $\mathcal{V} = Gr_G(\mathcal{W})$, $\mathcal{W} = CMon, Ab, R\text{-}Mod$ and G a group. Then:

$\mathcal{A}$ is α-accessible $\iff$ { $\mathcal{A}$ is conically α-accessible and has finite direct sums and (de)suspensions }

Prop: [Let $\mathcal{V}_0$ be cartesian closed and such that $\mathcal{V}_0(1,-): \mathcal{V}_0 \to \underline{Set}$ preserves coproducts and regular epimorphisms. Then

$\mathcal{A}$ is α-accessible $\iff$ $\mathcal{A}$ is conically α-accessible

Examples: Set, Cat, sSet, Gpd, Ord, pOrd, RGra

So far we had:

$\mathcal{A}$ α-accessible $\iff$ { $\mathcal{A}$ is conically α-accessible and Cauchy complete }

Is this true in general? (I don't know)

4) The characterization theorem

I) $\mathcal{V} = \underline{Set}$, cocomplete case

Theorem (Gabriel-Ulmer): the following are equivalent
1) $\mathcal{A}$ is locally presentable = (accessible + cocomplete)
2) $\mathcal{A} \simeq \alpha\text{-}Cont(\mathcal{C}, \underline{Set})$ for some α and $\mathcal{C}$ α-complete; $(\mathcal{C} = \mathcal{A}_\alpha^{op})$
3) $\mathcal{A} \simeq Mod(\mathcal{S}) \subseteq [\mathcal{C}, \underline{Set}]$ for a limit sketch $\mathcal{S} = (\mathcal{C}, \mathbb{L})$
4) $\mathcal{A}$ is a orthogonality class in some $[\mathcal{C}, \underline{Set}]$;
5) $\mathcal{A} \underset{\perp}{\hookrightarrow} [\mathcal{C}, \underline{Set}]$ accessibly embedded and reflective

(4) { $A \in \mathcal{K}$ is orthogonal with respect to $f: X \to Y$ in $\mathcal{K}$ if:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\forall"'] & Y \arrow[dl, dashed, "\exists!"] \\
A &
\end{tikzcd}$$

$\mathcal{K}(f,A): \mathcal{K}(Y,A) \to \mathcal{K}(X,A)$ is an iso (in $\mathcal{V}_0$) }

II) $\mathcal{V}$ general, cocomplete case (locally α_0-pres. as a closed)

Theorem (Kelly): the following are equivalent ($\alpha \geq \alpha_0$)
1) $\mathcal{A}$ is locally presentable; (accessible + cocomplete)
2) $\mathcal{A} \simeq \alpha\text{-}Cont(\mathcal{C}, \mathcal{V})$ for some α and $\mathcal{C}$ α-complete;
3) $\mathcal{A} \simeq Mod(\mathcal{S}) \subseteq [\mathcal{C}, \mathcal{V}]$ for a limit sketch $\mathcal{S} = (\mathcal{C}, \mathbb{L})$
4) $\mathcal{A}$ is a orthogonality class in some $[\mathcal{C}, \mathcal{V}]$;
5) $\mathcal{A} \underset{\perp}{\hookrightarrow} [\mathcal{C}, \mathcal{V}]$ accessibly embedded and reflective

III) $\mathcal{V} = \underline{Set}$, non cocomplete case

Theorem (Adámek, Rosický, Lair, Makkai, Paré): TFAE
1) $\mathcal{A}$ is accessible;
2) $\mathcal{A} \simeq \alpha\text{-}Flat(\mathcal{C}, \underline{Set})$ for some α and $\mathcal{C}$;
3) $\mathcal{A} \simeq Mod(\mathcal{S}) \subseteq [\mathcal{C}, \underline{Set}]$ for a limit/colimit sketch $\mathcal{S} = (\mathcal{C}, \mathbb{L}, \mathbb{C})$
4) $\mathcal{A}$ is a small cone-injectivity class;
5) $\mathcal{A}$ is accessibly embedded and cone-reflective in some $[\mathcal{C}, \underline{Set}]$.

(4) { $A \in \mathcal{K} = [\mathcal{C}, Set]$ is injective w.r.t the cone $(X \xrightarrow{f_i} Y_i)_{i \in I}$ if

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] \arrow[d] \arrow[ddr, bend right, "\forall"'] & Y_i \\
Y_j & Y_k \arrow[d, dashed, "\exists"] \\
 & A
\end{tikzcd}$$

$\coprod_i \mathcal{K}(Y_i, A) \xrightarrow{epi} \mathcal{K}(X,A)$ }

(5) {
$$\begin{tikzcd}
Ps^+_\mathcal{A} \subseteq [\mathcal{A}, Set]^{op} & \\
\mathcal{A} \arrow[u, hook] \arrow[r, hook] & {[\mathcal{C}, \underline{Set}]} \arrow[ul, dashed]
\end{tikzcd}$$

accessibly embedded
+
$\forall X \in [\mathcal{C}, Set]$ there is $\sum_i \mathcal{A}(A_i, -) \xrightarrow{epi} [\mathcal{C}, Set](X, J-)$ }

IV) $\mathcal{V}$ general, non cocomplete case

Theorem (B.Q.R): the following are equivalent
1) $\mathcal{A}$ is accessible
2) $\mathcal{A} \simeq \alpha\text{-}Flat(\mathcal{C}, \mathcal{V})$ for some α and $\mathcal{C}$;
3) $\mathcal{A} \simeq Mod(\mathcal{S}) \subseteq [\mathcal{C}, \mathcal{V}]$ for a limit/colimit sketch $\mathcal{S} = (\mathcal{C}, \mathbb{L}, \mathbb{C})$

Theorem (Lack, T.): the following are equivalent
1) $\mathcal{A}$ is accessible;
2) $\mathcal{A}$ is a virtual orthogonality class in some $[\mathcal{C}, \mathcal{V}]$;
3) $\mathcal{A}$ is accessibly embedded and virtually reflective in some $[\mathcal{C}, \mathcal{V}]$

(2) { $\mathcal{A} \hookrightarrow [\mathcal{C}, \mathcal{V}] =: \mathcal{K}$, consider the free completion $P^+\mathcal{K}$ of $\mathcal{K}$ with inclusion $Z: \mathcal{K} \hookrightarrow P^+\mathcal{K}$. $\left(P^+\mathcal{K} \xrightarrow{cont} \mathcal{L} \iff \mathcal{K} \to \mathcal{L} \right)$

consider orthogonality of $A \in \mathcal{K}$ w.r.to $ZX \xrightarrow{f} Y$ in $P^+\mathcal{K}$ }

(3) {
$$\begin{tikzcd}
P^+\mathcal{A} & \\
\mathcal{A} \arrow[u] \arrow[r, hook, "J"'] & {[\mathcal{C}, \mathcal{V}]} \arrow[ul, bend right, "L"']
\end{tikzcd}$$

$LX = \mathcal{K}(X, J-) \in P^+\mathcal{A} \subseteq [\mathcal{A}, \mathcal{V}]^{op}$ }

Theorem (Lack, T.): Let $\mathcal{A} \hookrightarrow [\mathcal{C}, \mathcal{V}]$; the following are equivalent:
1) $\mathcal{A}$ is accessible and accessibly embedded in $[\mathcal{C}, \mathcal{V}]$
2) $\mathcal{A}$ is a virtual orthogonality class in $\underline{[\mathcal{C}, \mathcal{V}]}$
3) $\mathcal{A}$ is accessibly embedded and virtually reflective in $[\mathcal{C}, \mathcal{V}]$.

Thank You!