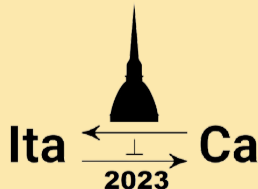


## On continuity of functors between locally presentable categories

Giacomo Tendas

18 December 2023



# The theorem

## Theorem

Let  $\mathcal{K}$  be locally finitely presentable. There exists a *regular cardinal*  $\gamma$  such that, given any finitary  $F: \mathcal{K} \rightarrow \mathcal{L}$ , with  $\mathcal{L}$  locally finitely presentable, the following are equivalent:

- ①  $F$  preserves *all  $\gamma$ -small limits*;
- ②  $F$  preserves *all small limits*;
- ③  $F$  has a left adjoint.

# Locally presentable categories

**Locally presentable categories** have been introduced by Gabriel and Ulmer in 1971, and have been thoroughly studied since then (Makkai–Paré, Adámek–Rosický, etc).

## Definition

A category  $\mathcal{K}$  is called **locally finitely presentable** if:

- $\mathcal{K}$  is cocomplete;
- has a set of **finitely presentable objects** which generate  $\mathcal{K}$  under filtered colimits;

For an infinitary version replace “finite” with “ $\lambda$ -small”, for a regular cardinal  $\lambda$ .

## These include:

- Finitary varieties;
- Finitary quasivarieties;
- Categories of models of finite limit sketches;
- Categories of models of essentially algebraic theories

# Examples

*The following are locally finitely presentable:*

- ①  $\mathcal{K} = \mathbf{Set}$
- ②  $\mathcal{K} = \mathbf{Pos}$
- ③  $\mathcal{K} = \mathbf{Bool}$
- ④  $\mathcal{K} = \mathbf{Gra}$
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- ⑧  $\mathcal{K} = \mathbf{Vect}$   $\rightarrow \mathcal{K}_f = \{\text{finite dimensional}\}.$



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( $R$  finite ring)
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See also a new preprint of Adámek and Sousa titled “A finitary adjoint functor theorem”.

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Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$  be a symmetric monoidal closed and locally finitely presentable category:

## Corollary

*There exists a regular cardinal  $\gamma$  such that an object  $X \in \mathcal{V}$  is dualizable if and only if*

$$X \otimes -: \mathcal{V} \longrightarrow \mathcal{V}$$

*is  $\gamma$ -continuous.*

# The enriched version

Fix  $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$  a symmetric monoidal closed and locally finitely presentable category. The results above also hold in the  $\mathcal{V}$ -enriched setting:

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## Small accessible $\mathcal{V}$ -categories

Introduced independently by Lair, Rosický, and Makkai–Paré:

$$\begin{aligned}
 \text{Accessible } (\mathcal{V}\text{-})\text{categories} &= \text{“locally presentable categories without} \\
 &\quad \text{cocompleteness”}; \\
 &= \mathcal{A} \text{ with filtered colimits + a set of finitely} \\
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 &= \text{Categories of models of limit/colimit sketches.}
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In the enriched context they were introduced by Borceux, Quinteiro, and Rosický.

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*A small  $\mathcal{V}$ -category is accessible if and only if it is Cauchy complete.*

**Thank You**