On continuity of functors between locally presentable categories

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18 December 2023

The theorem

Theorem

Let $\mathcal{K}$ be locally finitely presentable. There exists a regular cardinal γ such that, given any finitary $F\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally finitely presentable, the following are equivalent:

1. *F preserves all γ-small limits;*
2. *F preserves all small limits;*
3. *F has a left adjoint.*

Locally presentable categories

Locally presentable categories have been introduced by Gabriel and Ulmer in 1971, and have been thoroughly studied since then (Makkai–Paré, Adámek–Rosický, etc).

Definition

A category $\mathcal{K}$ is called **locally finitely presentable** if:

- $\mathcal{K}$ is cocomplete;
- has a set of finitely presentable objects which generate $\mathcal{K}$ under filtered colimits;

For an infinitary version replace "finite" with "λ-small", for a regular cardinal λ.

These include:

- Finitary varieties;
- Finitary quasivarieties;
- Categories of models of finite limit sketches;
- Categories of models of essentially algebraic theories

Examples

The following are locally finitely presentable:

1. $\mathcal{K} = \mathbf{Set}$
2. $\mathcal{K} = \mathbf{Pos}$
3. $\mathcal{K} = \mathbf{Bool}$
4. $\mathcal{K} = \mathbf{Gra}$
5. $\mathcal{K} = \mathbf{Cat}$
6. $\mathcal{K} = \mathbf{Ab}$
7. $\mathcal{K} = R\text{-}\mathbf{Mod}$
8. $\mathcal{K} = \mathbf{Vect}$

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The following are locally finitely presentable:

1. $\mathcal{K} = \mathbf{Set}$ $\quad\rightarrow \mathcal{K}_f = \{\text{finite sets}\};$
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5. $\mathcal{K} = \mathbf{Cat}$ $\rightarrow$ $\mathcal{K}_f = \{\text{finite obj.} + \text{finite set of generating mor.}\}$;
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7. $\mathcal{K} = R\text{-}\mathbf{Mod}$ $\rightarrow$ $\mathcal{K}_f = \{\text{finitely presented modules}\}$;
8. $\mathcal{K} = \mathbf{Vect}$ $\rightarrow$ $\mathcal{K}_f = \{\text{finite dimensional}\}$.

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The optimal regular cardinal γ provided by our proofs is one for which:

- $\mathcal{K}_f$ has less than γ objects (up to isomorphism);
- there exists β such that $\gamma > \beta > \#\mathcal{K}(X, Y)$ for each $X, Y \in \mathcal{K}_f$.

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See also a new preprint of Adámek and Sousa titled "A finitary adjoint functor theorem".

Dualizable objects

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Corollary

An R-module M is dualizable if and only if it is flat and

$$M \otimes - \colon R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$$

preserves γ-small products.

Note: if R is finite, $\gamma = \aleph_1$;

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Note: if R is finite, $\gamma = \aleph_1$;

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a symmetric monoidal closed and locally finitely presentable category:

Corollary

There exists a regular cardinal γ such that an object $X \in \mathcal{V}$ is dualizable if and only if

$$X \otimes - \colon \mathcal{V} \longrightarrow \mathcal{V}$$

is γ-continuous.

The enriched version

Fix $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ a symmetric monoidal closed and locally finitely presentable category. The results above also hold in the $\mathcal{V}$-enriched setting:

Theorem

Let $\mathcal{K}$ a locally finitely presentable $\mathcal{V}$-category. There exists a regular cardinal γ such that, given any finitary $F\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally finitely presentable, the following are equivalent:

1. *F preserves all γ-small weighted limits;*
2. *F preserves all small weighted limits;*
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The optimal regular cardinal γ provided by our proofs is one for which:

- $(\mathcal{K}_0)_f$ has less than γ objects (up to isomorphism);
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Small accessible $\mathcal{V}$-categories

Introduced independently by Lair, Rosický, and Makkai–Paré:

Accessible ($\mathcal{V}$-)categories	=	*"locally presentable categories without cocompleteness";*
	=	*$\mathcal{A}$ with filtered colimits + a set of finitely presentable objects which generate $\mathcal{A}$ under filtered colimits;*
	=	*Categories of models of limit/colimit sketches.*

In the enriched context they were introduced by Borceux, Quinteiro, and Rosický.

Cauchy (absolute) limits $=$ *Limits preserved by any $\mathcal{V}$-functor.*

Then:

Small accessible $\mathcal{V}$-categories

Introduced independently by Lair, Rosický, and Makkai–Paré:

$$\text{Accessible } (\mathcal{V}\text{-})\text{categories} = \textit{“locally presentable categories without cocompleteness”};$$

$$= \mathcal{A} \textit{ with filtered colimits + a set of finitely presentable objects which generate } \mathcal{A} \textit{ under filtered colimits;}$$

$$= \textit{Categories of models of limit/colimit sketches.}$$

In the enriched context they were introduced by Borceux, Quinteiro, and Rosický.

$$\text{Cauchy (absolute) limits} = \textit{Limits preserved by any } \mathcal{V}\textit{-functor.}$$

Then:

Corollary

A small $\mathcal{V}$-category is accessible if and only if it is Cauchy complete.

Thank You