CT 2025 - Brno -

Enriched Categorical Logic and Accessibility

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0) Background:

	Fragment of Logic	Cats of models	Cat notion of theory
equational:	$(\sigma = \tau)$	Fin. Varieties	Lawvere / cat with fin. prod.
cartesian:	sequents of $(\sigma=\tau), R(\tau), \wedge, \exists!$	Loc. fin. Pres.	Lex categories
regular:	sequents of $(\sigma=\tau), R(\tau), \wedge, \exists$	finite inject. class	Regular/exact cats
coherent:	sequents of $(\sigma=\tau), R(\tau), \wedge, \vee, \exists$	finite cover-ing classes.	Pretopoi

0) Background:

	Fragment of Logic	Cats of models	Cat notion of theory
equational:	$(\sigma = \tau)$	Fin. Varieties $Fp(\mathcal{C}, Set)$	Lawvere / cat with fin. prod.
cartesian:	$(\sigma=\tau), R(\tau), \wedge, \exists!$ sequents of	Loc. fin. Pres. $Lex(\mathcal{C}, Set)$	Lex categories
regular:	$(\sigma=\tau), R(\tau), \wedge, \exists$ sequents of	finite inject. class $Reg(\mathcal{C}, Set)$	Regular / exact cats
coherent:	$(\sigma=\tau), R(\tau), \wedge, \vee, \exists$ sequents of	finite cone-ing classes. $Mod(\mathcal{C}, Set)$	Pretopoi

0) Background: Infinitary case

	Fragment of Logic	Cats of models	Cat notion of theory
equational:	$(\sigma = \tau)$	(Fin.) Varieties	Lawvere / cat with fin. prod. (λ-small)
cartesian:	$(\sigma=\tau), R(\tau), \wedge, \exists!$ sequents of	Loc. (fin.) Pres.	Lex categories λ-small
regular:	$(\sigma=\tau), R(\tau), \wedge, \exists$ sequents of	(finite) inject. class = accessible cats with products	λ-Regular / exact cats
coherent:	$(\sigma=\tau), R(\tau), \wedge, \vee, \exists$ sequents of	(finite) cove-ing classes. = accessible cats	λ-Pretopoi

0) Background:

Enrichment

	Enriched Fragment of Logic	$\mathcal{V}$-Cats of models	$\mathcal{V}$-Cat notion of theory	
equational:	?	Fin. Varieties	$\mathcal{V}$-Cats with fin powers	(Power)
cartesian:	?	Loc. fin. Pres.	Lex $\mathcal{V}$-cats	(Kelly)
regular:	?	finite $\mathcal{E}$-injectivity classes (Lack-Rosický)	Regular / exact $\mathcal{V}$-cats	(Garner Lack)
coherent:	?	?	?	

0) Background:

Enrichment

	Enriched Fragment of Logic	V-Cats of models	V-Cat notion of theory
equational:	$(\sigma = \tau)$ CT2024 ↗	Fin. Varieties	V-Cats with fin powers
cartesian:	?	Loc. fin. Pres.	Lex V-cats
regular:	?	finite $\mathcal{E}$-injectivity classes	Regular / exact V-cats
coherent:	?	?	?

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	Enriched Fragment of Logic	$\mathcal{V}$-Cats of models	$\mathcal{V}$-Cat notion of theory
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	Enriched Fragment of Logic	$\mathcal{V}$-Cats of models	$\mathcal{V}$-Cat notion of theory
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1) Interpretation:

Fix a sufficiently good base of enrichment $\mathcal{V} = (\mathcal{V}_0, \otimes, I, [-,-])$ t.w. a factorization system $(\mathcal{E}, \mathcal{M})$

Examples:

- $\mathcal{V}$ regular category (e.g. Ab, DGAb, sSet, ...)

  $(\mathcal{E}, \mathcal{M}) = ($regular epi, monomorphism$)$

- $\mathcal{V} =$ Pos, Met, Cat

  $(\mathcal{E}, \mathcal{M}) = ($surjections, embed./isometry/ioo+ff.$)$

- $\mathcal{V} =$ Met, Ban

  $(\mathcal{E}, \mathcal{M}) = ($dense, closed isometry$)$

- $\mathcal{V} = \omega$-CPO

  $(\mathcal{E}, \mathcal{M}) = ($dense, closed embedding$)$

(v) Arities:

$n \in \mathbb{N} = \mathrm{FinSet} \subseteq \mathrm{Set}$

$\varphi(x_1, \ldots, x_n)$, $A \in \mathrm{Str}(\Sigma)$

$$\varphi_A \underset{\text{mono}}{\rightarrowtail} A^n \quad \text{in } \underline{\mathrm{Set}}$$

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$X \in V_f \subseteq V$ (or V_λ)

$\varphi(x)$, $x : X$, $A \in \mathrm{Str}(\mathbb{L})$

$$\varphi_A \underset{\mathcal{M}}{\rightarrowtail} A^X$$

(v) Arities:

$$\left.\begin{array}{l} n \in \mathbb{N} = \mathrm{FinSet} \subseteq \mathrm{Set} \\[1em] \varphi(x_1, \ldots, x_n) \;, \quad A \in \mathrm{Str}(\Sigma) \\[1em] \varphi_A \underset{\text{mono}}{\rightarrowtail} A^n \quad \text{in } \underline{\mathrm{Set}} \end{array}\right\} \longrightarrow \begin{array}{l} X \in \mathcal{V}_f \subseteq \mathcal{V} \quad (\text{or } \mathcal{V}_\lambda) \\[1em] \varphi(x),\; x : X \;, \quad A \in \mathrm{Str}(\mathbb{L}) \\[1em] \varphi_A \underset{\mathcal{M}}{\rightarrowtail} A^X \end{array}$$

<u>Examples</u>:

$\mathcal{V} = \mathrm{Pos}$, $\quad X = \{\cdot \leq \cdot\}$ $\qquad \varphi_A \subseteq \{(a,b) \in A \times A \mid a \leq b\} = A^X$

$\mathcal{V} = \mathrm{Met}$, $\quad X = [0,1]$ $\qquad \varphi_A \subseteq \{\gamma : [0,1] \longrightarrow A\} = A^X$

$\mathcal{V} = \mathrm{Cat}$, $\quad X = \{\cdot \leftarrow \cdot \rightarrow \cdot\}$ $\qquad \varphi_A \subseteq A^{\cdot \leftarrow \cdot \rightarrow \cdot} =$ "Spans in A"

$\mathcal{V} = \mathrm{Ban}$, $\quad X = (\mathbb{C}, t|\cdot|)$ $\qquad \varphi_A \subseteq A^X =$ "$\{x \in A \mid \|x\| \leq t\}$"

(ii) Formulas $\qquad\qquad\qquad\qquad$ $A \in \mathrm{Str}(\mathbb{L})$

a) $(\sigma = \tau) \quad \rightsquigarrow$

$$\begin{tikzcd} (\sigma=\tau)_A \arrow[r, tail] & A^X \arrow[r, shift left, "\sigma_A"] \arrow[r, shift right, "\tau_A"'] & A^Y \end{tikzcd}$$

b) $R(\tau) \quad \rightsquigarrow$

$$\begin{tikzcd} R(\tau)_A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^Y \arrow[d, "\tau_A"] \\ R_A \arrow[r, tail] & A^X \end{tikzcd}$$

c) $\varphi \wedge \psi \quad \rightsquigarrow$

$$\begin{tikzcd} (\varphi \wedge \psi)_A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varphi_A \arrow[d, tail] \\ \psi_A \arrow[r, tail] & A^X \end{tikzcd}$$

d) $\exists y\, \varphi(x,y) \quad \rightsquigarrow$

$$\begin{tikzcd} \varphi_A \arrow[r, tail] & A^{X+Y} \end{tikzcd}$$

(ii) Formulas $\qquad\qquad\qquad\qquad A \in \mathrm{Str}(\mathbb{L})$

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$$

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(\varphi\wedge\psi)_A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varphi_A \arrow[d, tail] \\
\psi_A \arrow[r, tail] & A^X
\end{tikzcd}
$$

d) $\exists y\, \varphi(x,y) \longmapsto$

$$
\begin{tikzcd}
 & (\exists y\, \varphi(x,y))_A \arrow[drr, tail, "\mathcal{M}"] & & \\
\varphi_A \arrow[ur, two heads, "\mathcal{E}"] \arrow[r, tail] & A^{X+Y} \arrow[rr, "\pi_1"'] & & A^X
\end{tikzcd}
$$

Examples

I) $\mathcal{V} = \underline{\mathbf{Met}}$

$$\varphi(x,y) := \exists\, p : [0,1] \quad (p(0) = x) \wedge (p(1) = y)$$

$(\mathcal{E}, \mathcal{M}) =$ (surjections , isometry)

$(\mathcal{E}, \mathcal{M}) =$ (dense, closed isometry)

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φ holds in $M \iff M$ is path connected

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Examples

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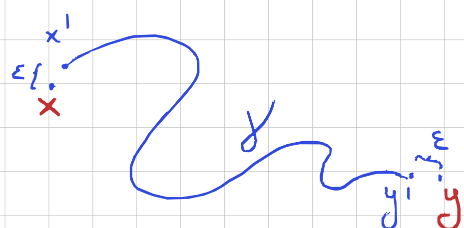
φ holds in M $\iff$ M is path connected

$$(\mathcal{E}, \mathcal{M}) = (\text{dense}, \text{closed isometry})$$

φ holds in M $\iff$ $\forall\, x, y \in M$ $\forall\, \varepsilon > 0$



Examples

II) $\mathcal{V} = \underline{\mathrm{Ban}}$

arities $\quad \mathbb{C} = (\mathbb{C}, |\cdot|)$, $t > 0$ $\quad \mathbb{C}_t = (\mathbb{C}, t\cdot|\cdot|) \in \underline{\mathrm{Ban}}$

idea: "$\forall x : \mathbb{C}_t, \ldots$" $\quad \leftrightsquigarrow \quad$ "$\forall x, \|x\| \leq t, \ldots$"

so we can express: "$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2$" by formulas

Examples

II) $\mathcal{V} = \underline{\mathrm{Ban}}$

arities $\mathbb{C} = (\mathbb{C}, |\cdot|)$, $t > 0$ $\mathbb{C}_t = (\mathbb{C}, t \cdot |\cdot|) \in \underline{\mathrm{Ban}}$

idea: " $\forall x : \mathbb{C}_t, \ldots$ " $\leftrightsquigarrow$ " $\forall x, \|x\| \leq t, \ldots$ "

so we can express: " $\|x+y\|^2 \leq \|x\|^2 + \|y\|^2$ " by formulas

$$\forall x : \mathbb{C}_p, \forall y : \mathbb{C}_q \;\; \exists z : \mathbb{C}_{\sqrt{p^2+q^2}} \;\; (z = x+y)$$

for $p, q \in \mathbb{Q}$ $(p^2+q^2 \leq 1)$

3) Presentation formulas

Given a language $\mathbb{L}$ and the $\mathcal{V}$-category $\mathrm{Str}(\mathbb{L})$ want to characterize the subcategories of the form

$$\mathrm{Mod}(\mathbb{T}) \hookrightarrow \mathrm{Str}(\mathbb{L}),$$

where $\mathbb{T}$ is given by a certain kind of sequents $\varphi \vdash \psi$.

$$\Big\}\ \text{need}$$
$$\downarrow$$

3) Presentation formulas

Given a language $\mathbb{L}$ and the $\mathcal{V}$-category $\mathrm{Str}(\mathbb{L})$ want to characterize the subcategories of the form

$$\mathrm{Mod}(\mathbb{T}) \hookrightarrow \mathrm{Str}(\mathbb{L}),$$

where $\mathbb{T}$ is given by a certain kind of sequents $\varphi \vdash \psi$.

$$\Big\downarrow \text{ need}$$

Say that $\varphi(x)$ presents $A \in \mathrm{Str}(\mathbb{L})$ if for any $B \in \mathrm{Str}(\mathbb{L})$

$$A \longrightarrow B \quad \iff \quad \begin{tikzcd} & \varphi_B \arrow[d, tail] \\ I \arrow[ur, dashed] \arrow[r, "\bar{b}"'] & B^X \end{tikzcd}$$

3) Presentation formulas

Given a language $\mathbb{L}$ and the $\mathcal{V}$-category $\mathrm{Str}(\mathbb{L})$ want to characterize the subcategories of the form

$$\mathrm{Mod}(\mathbb{T}) \hookrightarrow \mathrm{Str}(\mathbb{L}),$$

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$$\Big\Downarrow \text{ need}$$

Say that $\varphi(x)$ presents $A \in \mathrm{Str}(\mathbb{L})$ if

$$\begin{aligned} \mathrm{Str}(\mathbb{L})(A,\_) \cong \varphi_{(\_)} : \mathrm{Str}(\mathbb{L}) &\longrightarrow \mathcal{V} \\ B &\longmapsto \varphi_B \end{aligned}$$

4) Not everything works as usual:

$$\exists y\,(\varphi(x) \wedge \psi(x,y)) \equiv \varphi(x) \wedge \exists y\, \psi(x,y)$$

- Frobenius rule -

holds ordinarily

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$$\exists y\,(\varphi(x) \wedge \psi(x,y)) \vdash \varphi(x) \wedge \exists y\, \psi(x,y)$$

- $\dashv$ need not hold (unless $\mathcal{E}$ is pullback stable)
- problem when doing substitution and nested exist. quantification.

This affected the notion of regular theory:

4) Not everything works as usual:

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This affected the notion of regular theory:

<u>Def</u>: A regular theory $\mathbb{T}$ is a set of sequents of the form

$$(\forall x)\quad \varphi(x) \vdash \exists y\,(\varphi(x) \wedge \psi(x,y))$$

with φ an ψ conjunctions of atomic formulas.

We prove:

<u>Theorem</u>: The following are equivalent for $\mathcal{A} \subseteq \mathrm{Str}(\mathbb{L})$:

i) $\mathcal{A} = \mathrm{Mod}(\mathbb{T})$ for a regular theory $\mathbb{T}$.

ii)

iii) $\mathcal{A} = \mathcal{H}\text{-inj}_{\mathcal{E}}$ is an $\mathcal{E}$-injectivity class

iv)

v)

In nicer cases, covering the examples mentioned before, we prove

Theorem: The following are equivalent for $\mathcal{A} \subseteq \mathrm{Str}(\mathbb{L})$:

i) $\mathcal{A} = \mathrm{Mod}(\mathbb{T})$ for a regular theory $\mathbb{T}$.

★ ii) $\mathcal{A} = \mathrm{Mod}(\mathbb{T})$, where the sequents in $\mathbb{T}$ are of the form

$$\exists y\ \varphi(x,y) \vdash \exists z\ \psi(x,z)$$

φ, ψ conj. of atomic formulas.

iii) $\mathcal{A} = \mathcal{H}\text{-}\mathrm{Inj}_{\mathcal{E}}$ is an $\mathcal{E}$-injectivity class.

★ iv) $\mathcal{A}$ is closed under products, powers by $\mathcal{E}$-stable objects, (λ-) filtered colimits, and (λ-) elementary subobjects.

★ v) $\mathcal{A}$ is accessible and closed under the constructions in (iv)

Conclusion:

we define fragments of logic that present $\mathcal{V}$-categories. See:

① Towards enriched universal algebra & On enriched terms and 2-dimensional universal algebra

② Enriched concepts of regular logic

③ Enriched positive logic (to appear)

	Enriched Fragment of Logic	$\mathcal{V}$-Cats of models
①	$(\sigma = \tau)$	Fin. Varieties
③	$(\sigma=\tau), R(\tau), \wedge, \exists!$ sequents of	Loc. fin. Pres.
②	$(\sigma=\tau), R(\tau), \wedge, \exists$ sequents of	finite $\mathcal{E}$-injectivity classes
③	$(\sigma=\tau), R(\tau), \wedge, \vee, \exists$ sequents of	finite cone $\mathcal{E}$-inj. classes