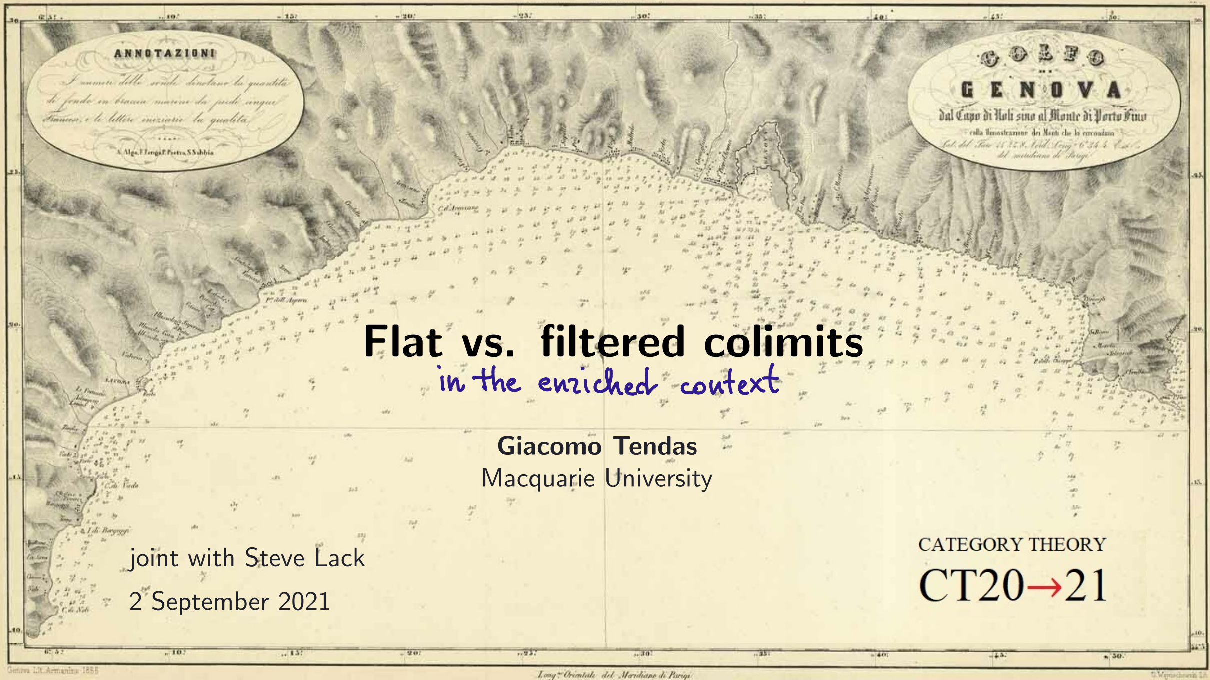

Flat vs. filtered colimits

in the enriched context

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CATEGORY THEORY
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Motivation: accessible categories

Some History:

- Ehresmann: Models of sketches (1968)
- Gabriel and Ulmer: locally presentable categories (1971)
- Lair and Makkai/Paré: Accessible categories (80's)

Examples:

- Presheaf categories;
- Grothendieck topoi;
- Locally presentable categories;
- Categories of models of sketches;
- Categories of models of first order theories.

$\Leftrightarrow$ Accessible

Properties:

- Have a small dense generator;
- complete iff cocomplete;
- stable in **CAT** under flexible limits; (pseudo/bilimits)
- adjoint functor theorems.

...

Two characterizations

Proposition

The following are equivalent for a category $\mathcal{A}$:

1. *$\mathcal{A}$ is the free cocompletion of a small category under filtered colimits;*
2. *$\mathcal{A}$ is equivalent to the category $\mathrm{Flat}(\mathcal{C}^{op}, \mathbf{Set})$ of flat presheaves on a small category $\mathcal{C}$.*

If either of those holds we say that $\mathcal{A}$ is **finitely accessible**.

- (1) is more convenient to work with:
- (2) is useful to develop the theory:

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If either of those holds we say that $\mathcal{A}$ is **finitely accessible**.

- (1) is more convenient to work with: $\mathcal{A}(A,-)$ finitary

  $\mathcal{A}$ is finitely accessible $\Leftrightarrow \exists\, \mathcal{C} \subseteq \mathcal{A}_f$ small such that every object of $\mathcal{A}$ is a filtered colimit of objects from $\mathcal{C}$.

- (2) is useful to develop the theory:

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- (1) is more convenient to work with:

$$\mathcal{A} \text{ is finitely accessible} \Leftrightarrow \exists\, \mathcal{C} \subseteq \mathcal{A}_f \text{ small such that every object of } \mathcal{A} \text{ is a filtered colimit of objects from } \mathcal{C}.$$

- (2) is useful to develop the theory:

$$\begin{aligned} \mathcal{A} \text{ is accessible} &\Leftrightarrow \mathcal{A} \text{ is the category of models of a sketch} \\ &\Leftrightarrow \mathcal{A} \text{ is the category of models of a first order theory.} \end{aligned}$$

Flatness

Proposition

Let $M : \mathcal{C}^{op} \to \mathbf{Set}$ be a functor; the following are equivalent:

1. *the category of elements* $\mathrm{El}(M)$ *is filtered;*
2. M *is a filtered colimit of representable functors;*
3. $\mathrm{Lan}_Y M \colon [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ *preserves finite limits.*

If any of those holds we say that F is a **flat** *functor.*

Important later:

$$M \cong \operatorname{colim}\left(\underbrace{\mathrm{El}(M)}_{\text{filtered}} \xrightarrow{\pi} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathbf{Set}]\right)$$

Enriched categories

Replace the sets of morphisms with objects of a symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$.

- A $\mathcal{V}$-enriched category $\mathcal{B}$ is given by:
  1. a collection of objects $X, Y, Z, \ldots$;
  2. morphism objects $\mathcal{B}(X, Y) \in \mathcal{V}$, for each X, Y;
  3. maps $\mathcal{B}(Y, Z) \otimes \mathcal{B}(X, Y) \to \mathcal{B}(X, Z)$ in $\mathcal{V}$;
  4. identities $\mathrm{id}_X : I \to \mathcal{B}(X, X)$.
- We can do ordinary category theory in this setting:

$F : \mathcal{C} \to \mathcal{D}$, , $[\mathcal{C}, \mathcal{B}]$, $Y : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$

$\eta : F \to G$,

Examples of $\mathcal{V}$:

- $(\textbf{Cat}, \times, 1)$;
- $(\textbf{Pos}, \times, 1)$;
- $(\textbf{SSet}, \times, 1)$;
- $(\textbf{Ab}, \otimes, I)$;
- $(\textbf{GAb}, \otimes, I)$;
- $(\textbf{DGAb}, \otimes, I)$;
- $([0, \infty], +, 0)$.

In general we assume that our base $\mathcal{V}$ is locally finitely presentable as a closed category:

$(\mathcal{V}_0)_f \ni I$, closed under $\otimes$

Kelly

Enriched accessibility

Definition

Let $\mathcal{A}$ be a $\mathcal{V}$-category; we say that:

1. $\mathcal{A}$ is **conically finitely accessible** if it is the free cocompletion of a small $\mathcal{V}$-category under filtered colimits;
2. $\mathcal{A}$ is **finitely accessible** if it is equivalent to the $\mathcal{V}$-category $\mathrm{Flat}(\mathcal{C}^{op}, \mathcal{V})$ of flat presheaves on a small $\mathcal{V}$-category $\mathcal{C}$.

- (1) good to work with, $\mathcal{V} = \mathrm{Cat}$, SSet, Ab
- (2) good for the theory $\rightsquigarrow$ Models of Sketches [BQR]
- (1) $\Leftrightarrow$ (2)? No
  ↳ Yes $\Leftrightarrow$ cocomplete

Flatness

Definition

We say that $M \colon \mathcal{C}^{op} \to \mathcal{V}$ is **flat** if its left Kan extension $\mathrm{Lan}_Y M \colon [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ along the Yoneda embedding preserves finite weighted limits. finite conical + finite powers

- **Aim:** Characterize flat $\mathcal{V}$-functors in terms of filtered colimits, when possible.
- **Strategy:** the ordinary functor $\mathcal{V}_0(I, -) \colon \mathcal{V}_0 \to \mathbf{Set}$ induces an adjunction:

$$\begin{tikzcd} {[\mathcal{C}^{op}, \mathcal{V}]_0} \arrow[rr, "\mathscr{U}"'] \arrow[rr, phantom, "\perp"] & & {[\mathcal{C}_0^{op}, \mathbf{Set}]} \arrow[ll, bend right, "\mathscr{F}"'] \end{tikzcd}$$

$$M \longmapsto \mathcal{C}_0^{op} \xrightarrow{M_0} \mathcal{V}_0 \xrightarrow{\mathcal{V}_0(I,-)} \mathrm{Set}$$

We give conditions so that:

(I) if M is flat, then $\mathscr{U} M$ is flat;

(II) if M is flat, then $\epsilon_M \colon \mathscr{F}\mathscr{U} M \to M$ is invertible.

When flat $=$ filtered

Assume that the unit I of $\mathcal{V}$ satisfies the following conditions:

(a) $\mathcal{V}_0(I,-)\colon \mathcal{V}_0 \to \mathbf{Set}$ is *cocontinuous*. (weakly)

(b) $\mathcal{V}_0(I,-)\colon \mathcal{V}_0 \to \mathbf{Set}$ is *strong monoidal*. (weakly)

$$(a)\quad \operatorname{colim} \mathcal{V}_0(I, H-) \xrightarrow{\not\cong} \mathcal{V}_0(I, \operatorname{colim} H)$$

$$(b)\quad \mathcal{V}_0(I,A) \times \mathcal{V}_0(I,B) \xrightarrow{\not\cong} \mathcal{V}_0(I, A \otimes B)$$

Examples of $\mathcal{V}$:

- $(\mathbf{Set}, \times, 1)$;
- $(\mathbb{2}, \times, 1)$;
- $(\mathbf{Cat}, \times, 1)$;
- $(\mathbf{SSet}, \times, 1)$;
- $(\mathbf{Pos}, \times, 1)$;
- $(\mathbf{Gpd}, \times, 1)$;
- $(2\text{-}\mathbf{Cat}, \boxtimes, 1)$;
- $(\mathbf{Met}, \otimes, 1)$;
- $(\mathcal{V}\text{-}\mathbf{Cat}, \otimes, \mathcal{I})$.
- $(\mathrm{Set}_*, \wedge, I)$

When flat = filtered

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(a) $\mathcal{V}_0(I,-)\colon \mathcal{V}_0 \to \mathbf{Set}$ is *cocontinuous*.

(b) $\mathcal{V}_0(I,-)\colon \mathcal{V}_0 \to \mathbf{Set}$ is *strong monoidal*.

Proposition

Let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a flat $\mathcal{V}$-functor. Then

$$M \cong \operatorname{colim}\left(\mathrm{El}(\mathscr{U}M)_{\mathcal{V}} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op},\mathcal{V}]\right) \cong \mathfrak{F}\mathfrak{U}M$$

and $\mathrm{El}(\mathscr{U}M)$ *is a filtered category.*

Theorem

A $\mathcal{V}$-category $\mathcal{A}$ is finitely-accessible if and only if it is conically finitely accessible.

Examples of $\mathcal{V}$:

- $(\mathbf{Set}, \times, 1)$;
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- $(\mathcal{V}\text{-}\mathbf{Cat}, \otimes, \mathcal{I})$.

When flat = filtered + absolute (1)

We say that $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is **locally dualizable** if:

(a) $\mathcal{V}_0$ has finite direct sums;

(b) The unit I is regular projective and finitely presentable;

(c) $\mathcal{V}_0$ has a strong generator $\mathcal{G}$ made of dualizable objects;

(d) for every arrow $z\colon I \to A \otimes B$ there exists a dualizable object $P \in \mathcal{V}$ and maps $x\colon P \to A$ and $y\colon P^* \to B$ such that

$$\begin{tikzcd} I \arrow[r, "\eta_P"] \arrow[dr, "z"'] & P \otimes P^* \arrow[d, "x \otimes y"] \\ & A \otimes B \end{tikzcd}$$

commutes.

Examples of $\mathcal{V}$:

- $(\mathbf{CMon}, \otimes, \mathbb{N})$ with $\mathcal{G} = \{\mathbb{N}\}$;
- $(\mathbf{Ab}, \otimes, \mathbb{Z})$ with $\mathcal{G} = \{\mathbb{Z}\}$;
- $(R\text{-}\mathbf{Mod}, \otimes, R)$ with $\mathcal{G} = \{R\}$;
- $(\mathbf{GAb}, \otimes, I)$ with $\mathcal{G} = \{S_n I\}_{n \in \mathbb{Z}}$;
- $(\mathbf{G}\text{-}\mathbf{Gr}(R\text{-}\mathbf{Mod}), \otimes, I)$ with $\mathcal{G} = \{S_g R\}_{g \in \mathbf{G}}$.

When flat = filtered + absolute (2)

Proposition

Let $\mathcal{C}$ be a $\mathcal{V}$-category with finite direct sums and copowers by dualizable objects, and let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a flat $\mathcal{V}$-functor. Then

$$M \cong \operatorname{colim}\left(\mathrm{El}(\mathscr{U} M)_{\mathcal{V}} \xrightarrow{\pi_{\mathcal{V}}} \mathcal{C} \xrightarrow{Y} [\mathcal{C}^{op}, \mathcal{V}]\right)$$

and $\mathrm{El}(\mathscr{U} M)$ *is a filtered category.*

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and $\mathrm{El}(\mathscr{U} M)$ is a filtered category.

Theorem

A $\mathcal{V}$-functor $M\colon \mathcal{C}^{op} \to \mathcal{V}$ is flat if and only if it is a filtered colimit of absolute colimits of representables.

Theorem

A $\mathcal{V}$-category has all absolute colimits if and only if it has

- finite direct sums;
- copowers by dualizable objects;
- splittings of idempotents.

Theorem

Let $\mathcal{A}$ be a $\mathcal{V}$-category; TFAE:

- $\mathcal{A}$ is finitely-accessible;
- $\mathcal{A}$ is conically finitely accessible and has all absolute colimits.

When flat = filtered + absolute (2)

Proposition

Let $\mathcal{C}$ be a $\mathcal{V}$-category with finite direct sums and copowers by dualizable objects, and let $M\colon \mathcal{C}^{op} \to \mathcal{V}$ be a flat $\mathcal{V}$-functor. Then

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Note: In general flat $\neq$ filtered + absolute. Take for example $\mathcal{V} = \mathbf{Set}^G$ for a finite non-trivial group G.

Thank You