Enriched universal algebra

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Overview

Ordinary universal algebra:

$\mathbb{L}$ language (signature) containing n-ary function symbols, where $n \in \mathbb{N} = \mathrm{Fin}(\mathbf{Set})$.

$\mathbb{L}$-structure: $A \in \mathbf{Set}$ t.w.

$$f_A \colon A^n \to A$$

for $f \in \mathbb{L}$.

Then there are $\mathbb{L}$-terms, equations, and models (algebras) for such equations.

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Enriched universal algebra

$\mathbb{L}$ language (signature) containing (X, Y)-ary function symbols, where $X \in \mathrm{Fin}(\mathcal{V})$.

$\mathbb{L}$-structure: $A \in \mathcal{V}$ t.w.

$$f_A \colon A^X \to A^Y$$

for $f \in \mathbb{L}$.

Then we define $\mathbb{L}$-terms, equations, and models (algebras) for such equations.

Here $\mathcal{V}$ is a category endowed with additional structure on it.

Motivation

Ordinarily, categories of models of equational theories are well understood from the categorical point of view.

Theorem (Lawvere, Linton)

The following are equivalent for a given category $\mathcal{K}$:

1. $\mathcal{K} = \mathrm{Mod}(\mathbb{T})$ *for an equational theory $\mathbb{T}$ on some $\mathbb{L}$;*
2. $\mathcal{K} = \mathrm{Alg}(T)$ *for a finitary monad $T \colon \mathbf{Set} \to \mathbf{Set}$;*
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For certain categories $\mathcal{V}$, one can consider $\mathcal{V}$-enriched categories $\mathcal{K}$:

- a set $\mathrm{Ob}(\mathcal{K})$ of objects;
- for any $A, B \in \mathrm{Ob}(\mathcal{K})$ an hom-object

$$\mathcal{K}(A, B)$$

in $\mathcal{V}$, together with...

- Many results from ordinary category theory have enriched analogues.

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Theorem (Power)

The following are equivalent for a given $\mathcal{V}$-enriched category $\mathcal{K}$:

1. $\mathcal{K} =$???
2. $\mathcal{K} = \mathrm{Alg}(T)$ *for a finitary monad $T\colon \mathcal{V} \to \mathcal{V}$;*
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(If you are not interested in enrichment: I want to do universal algebra internal to $\mathcal{V}$.)

Setting

Our base $\mathcal{V}$ is a symmetric monoidal closed category (etc.):

- There is a unit $I \in \mathcal{V}$ and, for any A, B we have
$$A \otimes B \in \mathcal{V}.$$
- These satisfy
$$A \otimes B \cong B \otimes A$$
and $A \otimes I \cong A$.
- For any $A, X \in \mathcal{V}$ there is $A^X \in \mathcal{V}$ s.t.
$$B \to A^X \iff B \otimes X \to A.$$
- There is a well defined notion of finite object in $\mathcal{V}$.
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- $\mathcal{V} = (\mathbf{Set}, \times, 1)$
- $\mathcal{V} = (\mathbf{Ab}, \otimes, \mathbb{Z})$
- $\mathcal{V} = (\mathbb{K}\text{-}\mathbf{Vect}, \otimes_{\mathbb{K}}, \mathbb{K})$
- $\mathcal{V} = (\mathbf{Pos}, \times, 1)$

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- $\mathcal{V} = (\mathbf{Set}, \times, 1)$

  Finite;
- $\mathcal{V} = (\mathbf{Ab}, \otimes, \mathbb{Z})$

  Finitely presented;
- $\mathcal{V} = (\mathbb{K}\text{-}\mathbf{Vect}, \otimes_{\mathbb{K}}, \mathbb{K})$

  Finite dimensional;
- $\mathcal{V} = (\mathbf{Pos}, \times, 1)$

  Finite.

Enriched languages

Definition

A single-sorted (functional) language $\mathbb{L}$ over $\mathcal{V}$ is the data of a set of function symbols $f\colon (X, Y)$ whose arities X and Y are finite objects of $\mathcal{V}$.

- For $\mathcal{V} = \mathbf{Set}$, finite sets are finite ordinals $n = \{0, 1, \ldots, n-1\} \in \mathbb{N}$.
- $(n, 1)$-ary $\iff$ n-ary function symbol.
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To define an $\mathbb{L}$-structure, we take $A \in \mathbf{Set}$ together with

- $f_A\colon A^n \to A$ if $f \in \mathbb{L}$ is n-ary
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- $(n, 1)$-ary $\iff$ n-ary function symbol.
- (n, m)-ary $\iff$ m-tuple $(f_i)_{i \leq m}$ of n-ary function symbols.

To define an $\mathbb{L}$-structure, we take $A \in \mathbf{Set}$ together with

- $f_A\colon A^n \to A$ if $f \in \mathbb{L}$ is n-ary
- $f_A\colon A^n \to A^m$ if $f = (f_i)_{i \leq m} \in \mathbb{L}$ is (n, m)-ary.

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Definition

Given a language $\mathbb{L}$, an $\mathbb{L}$-structure is the data of an object $A \in \mathcal{V}$ together with a morphism

$$f_A\colon A^X \to A^Y$$

in $\mathcal{V}$ for any function symbol $f\colon (X, Y)$ in $\mathbb{L}$.

A morphism of $\mathbb{L}$-structures $h\colon A \to B$ is the data of a map $h\colon A \to B$ in $\mathcal{V}$ making the following square commute for any $f\colon (X, Y)$ in $\mathbb{L}$.

$$\begin{array}{ccc} A^X & \xrightarrow{f_A} & A^Y \\ {\scriptstyle h^X}\downarrow & & \downarrow{\scriptstyle h^Y} \\ B^X & \xrightarrow[f_B]{} & B^Y \end{array}$$

Terms

Definition

The class of $\mathbb{L}$-terms is defined recursively as follows:

1. Every morphism $g\colon Y \to X$ of finite objects is an (X, Y)-ary term;
2. Every function symbol $f : (X, Y)$ of $\mathbb{L}$ is an (X, Y)-ary term;
3. if t_i is an (X_i, Y_i)-ary term for $i \leq n$, and s is an $(\sum_{i \leq n} Y_i, W)$-ary term; then
$$s(t_1, \ldots, t_n)$$
is a $(\sum_{i \leq n} X_j, W)$-ary term;
4. If t is a (X, Y)-ary term and Z is finite, then t^Z is a $(Z \otimes X, Z \otimes Y)$-ary term.

When $\mathcal{V} = \mathbf{Set}$, an (n, m)-ary term is a m-tuple of n-ary terms.

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Given $k\colon 1 \to n$, the n-ary term corresponding to it is the k-th projection $\pi_k(x_1, \dots, x_n)$.

Given A an $\mathbb{L}$-structure, the interpretation of $g : (X, Y)$ is
$$g_A := A^g\colon A^X \to A^Y$$
precomposition by g.

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Needs no explanation:
function symbols are terms.

Given A an $\mathbb{L}$-structure, the interpretation of $f : (X, Y)$ is

$$f_A \colon A^X \to A^Y$$

(part of the structure on A).

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Defines superposition of terms: if $Y_i = 1$ this is standard superposition. Otherwise is (pointwise) superposition of tuples.

Given A an $\mathbb{L}$-structure, the interpretation of $s(t_1, \ldots, t_n)$ is

$$A^{\Sigma_i X_i} \xrightarrow{\prod_i (t_i)_A} A^{\Sigma_i Y_i} \xrightarrow{s_A} A^W$$

where $A^{\Sigma_i X_i} \cong \prod_i A^{X_i}$.

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When $\mathcal{V} = \mathbf{Set}$, an (n, m)-ary term is a m-tuple of n-ary terms.

Given an n-ary term t defines an (nm, m)-ary term t^m which corresponds to the tuple $(t, \dots, t)$.

Given A an $\mathbb{L}$-structure, the interpretation of t^Z is

$$A^{Z \otimes X} \xrightarrow{(t_A)^Z} A^{Z \otimes Y}$$

where $A^{Z \otimes X} \cong (A^X)^Z$.

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Enriched equational theories

Definition

Given two $\mathbb{L}$-terms s, t of arity (X, Y) we can form the equation

$$(s = t).$$

We say that an $\mathbb{L}$-structure A satisfies $(s = t)$ if $s_A = t_A$. A set $\mathbb{E}$ of equations is called an equational theory. We denote by $\mathrm{Mod}(\mathbb{E})$ the (enriched) category of models of $\mathbb{E}$.

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Theorem

The following are equivalent for an enriched category $\mathcal{K}$:

1. $\mathcal{K} \simeq \mathrm{Mod}(\mathbb{E})$ *for some equational theory* $\mathbb{E}$*;*
2. $\mathcal{K} \simeq \mathrm{Alg}(T)$ *for a finitary monad* T *on* $\mathcal{V}$*;*
3. $\mathcal{K} \simeq \mathrm{FP}(\mathcal{T}, \mathcal{V})$ *is the category of models of an enriched Lawvere theory.*

Examples: $\mathbb{K}$-Vect and Ab

$\mathcal{V} = \mathbb{K}$-**Vect**

$\mathrm{Fin}(\mathbb{K}\text{-}\mathbf{Vect}) = \{\mathbb{K}^n\}_{n\geq 0}$ and

$$V^{\mathbb{K}^n} \cong V^n$$

for any $\mathbb{K}$-vector space V.
A language $\mathbb{L}$ over $\mathbb{K}$-**Vect** is just an ordinary language. An $\mathbb{L}$-structure is given by $V \in \mathbb{K}$-**Vect** t.w.

$$f_V \colon V^n \to V$$

for any $f \in \mathbb{L}$.
Terms are build as ordinary terms plus:

- if s, t are terms, then $s + t$ is a term;
- if t is a term and $k \in \mathbb{K}$, then $k \cdot t$ is a term.

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$\mathcal{V} = \mathbf{Ab}$

$\mathrm{Fin}(\mathbf{Ab})$ = direct sums of $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$; and

$$G^{\mathbb{Z}/m\mathbb{Z}} \cong G_m := \{x \in G | mx = 0\}$$

for any abelian group G.
A language $\mathbb{L}$ has function sym. with arity

$$(\mathbb{Z}^n \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z},\ \mathbb{Z}/m\mathbb{Z}),$$

an $\mathbb{L}$-structure is given by $G \in \mathbf{Ab}$ t.w.

$$f_G \colon G^n \oplus G_{m_1} \oplus \cdots \oplus G_{m_k} \to G_m$$

for any $f \in \mathbb{L}$.
Terms are build as ordinary terms plus:

- if s, t are terms, then $s + t$ is a term and $-t$ is a term.

Examples: Pos and Met

$\mathcal{V} = \mathbf{Pos}$

$\mathrm{Fin}(\mathbf{Pos}) = \{X = (S, \leq) \mid S \text{ is finite}\}$ and

$$P^X \cong \mathbf{Pos}(X, P)$$

for any poset P.
Consider $\mathbb{2} := \{0 \leq 1\}$, so that

$$P^{\mathbb{2}} = \{(x, y) \in P \times P \mid x \leq y\}.$$

It is enough to work with $(X, \mathbb{2})$-ary terms (believe me).
Suppose $X = n$ is discrete. Any $(n, \mathbb{2})$-ary term t, with interpretation $t_P \colon P^n \to P^{\mathbb{2}}$, induces two classical n-ary terms t^1, t^2 such that

$$t^1_P(a_1, \dots a_n) \leq t^2_P(a_1, \dots, a_n).$$

Thus, equational theories over **Pos** include inequalities. See [Adámek, Ford, Milius, Schröder].

Examples: Pos and Met

$\mathcal{V} =$ **Met**

Metric spaces with non-expanding maps, consider a countable metric space X

$$M^X \cong \mathbf{Met}(X, M)$$

for any metric space M.
Consider $\mathcal{2}_\varepsilon := \{0, 1\}$ with $d(0, 1) = \varepsilon$, for $\varepsilon > 0$, so that

$$M^{\mathcal{2}_\varepsilon} = \{(x, y) \in P \times P \mid d(x, y) \leq \varepsilon\}.$$

It is enough to work with $(X, \mathcal{2}_\varepsilon)$-ary terms (believe me).
Suppose $X = n$ is discrete. Any $(n, \mathcal{2}_\varepsilon)$-ary term t, with interpretation $t_M \colon M^n \to M^{\mathcal{2}_\varepsilon}$, induces two classical n-ary terms t^1, t^2 such that

$$d(t^1_M(a_1, \ldots a_n), t^2_M(a_1, \ldots, a_n)) \leq \varepsilon.$$

Thus, equational theories over **Met** include equalities up to ε. See [Adámek, Dostál, Velebil].

Thank You