On continuity of functors between locally presentable categories

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Locally presentable categories

Locally presentable categories have been introduced by Gabriel and Ulmer in 1971.

Examples include:

- (Finitary) varieties; ← models of equational theories (Grp, Rng ...)
- (Finitary) quasivarieties; ← " " implication theories (Gra, Ab_{tf})
- Grothendieck topoi;
- Categories of models of limit sketches; ⟺ locally pres.
- Categories of models of essential algebraic theories

Locally finitely presentable categories

Definition by examples:

- Set : finite Set $\quad X \in \text{Set}$ then $X = \bigcup_{\substack{X_i \subseteq X \\ \text{finite}}} X_i$ $\quad \left(\begin{matrix} X_i \\ X_j \end{matrix} \rightrightarrows X_{ij} \right)$
- R-Mod : fin. presented $\quad M \in R\text{-Mod}$ then $M = \bigcup_{\substack{M_i \leq M \\ \text{f.p}}} M_i$ $\quad \left(\begin{matrix} M_i \\ M_j \end{matrix} \rightrightarrows M_{ij} \right)$
- Pos : finite posets : - - - - -

  part. ordered set

The actual definition

has all coproducts and coequalizers.

Definition

A category $\mathcal{K}$ is called **locally finitely presentable** if it is cocomplete and has a set of finitely presentable objects which generates $\mathcal{K}$ under directed colimits.

$X \in K_f$ if $\mathrm{Hom}(X, -) : K \longrightarrow \underline{Set}$ preserves directed colimits

every object of K is a directed colimit of objects from K_f.

Other examples:

$\underline{GAb}$, $\underline{DGAb}$, $\underline{Cat}$, $\underline{SSet}$

$\underline{Gra}$, $\underline{Ab}_{tf}$, ...

Infinitary version:

replace "finite" with "less than α"

$\underline{Met}$, $\underline{Ban}$, ...

The theorem

F preserves directed colimits

Theorem

Let $\mathcal{K}$ be locally finitely presentable. There exists a regular cardinal γ for which every finitary $F\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally finitely presentable, is γ-continuous if and only if it is continuous.

F preserves γ-small products and equalizers. $\Longrightarrow$ F preserves products and equalizers.

The optimal regular cardinal γ provided by our proofs is one for which:

- $\mathcal{K}_f$ has less than γ objects (up to isomorphism);
- there exists β such that $\gamma > \beta > \#\mathrm{Hom}(X, Y)$ for each $X, Y \in \mathcal{K}_f$.

eg. $K = \mathrm{Set}$ $\quad \gamma = \aleph_1$

Dualizable objects

fin. gen + proj.

An R-module M is called dualizable if $M \otimes - \cong \mathrm{Hom}(M^*, -)$ for some $M^* \cong \mathrm{Hom}(M, I)$

Corollary

An R-module M is dualizable if and only if it is flat and $M \otimes -$ preserves γ-small products.

Note: if R if finite, then R-Mod_f has countably many objects and its hom-sets are all finite; so we can choose $\gamma = \aleph_1$;

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Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a symmetric monoidal closed and locally presentable category:

Corollary

There exists a regular cardinal γ such that an object $X \in \mathcal{V}$ is dualizable if and only if $X \otimes - : \mathcal{V} \to \mathcal{V}$ is γ-continuous.

An adjoint functor theorem

In general, for finitely presentable categories we know that:

Theorem

For any finitary functor $U\colon \mathcal{K} \to \mathcal{L}$ between locally finitely presentable categories, the following are equivalent:

1. *U has a left adjoint;*
2. *U is continuous.*

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Fixing the domain $\mathcal{K}$, we can specialize it as follows:

Theorem

Let $\mathcal{K}$ be locally finitely presentable. There exists a regular cardinal γ such that for any finitary functor $U\colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally finitely presentable, the following are equivalent:

1. *U has a left adjoint;*
2. *U is γ-continuous.*

The enriched version

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a symmetric monoidal closed and locally finitely presentable category. The results above also hold in an enriched setting:

Theorem

Let $\mathcal{K}$ be a locally finitely presentable $\mathcal{V}$-category. There exists a regular cardinal γ for which every finitary $F \colon \mathcal{K} \to \mathcal{L}$, with $\mathcal{L}$ locally finitely presentable, is γ-continuous if and only if it is continuous.

My initial motivation was to prove:

has all limits preserved by any functor

Theorem

A small $\mathcal{V}$-category is accessible if and only if it is Cauchy complete.

Thank You