Regular Theories Enriched Over Finitary Varieties

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Joint work with:
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Outline

Theories in Logic

A Theory is given by a list of axioms on a fixed set of operations; its models are corresponding sets and functions that satisfy those axioms.

Algebraic Theories:

- Axioms consist of equations based on the operation symbols of the language;

Abelian groups:

Models of the following algebraic theory:

$$(x+y)+z = x+(y+z), \quad x+y = y+x$$
$$x+0 = x, \quad x+(-x) = 0,$$

With operation symbols $+ : G \times G \to G$ and $-(\ ) : G \to G$.

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Essentially Algebraic Theories:

- Axioms are still equations but the operation symbols are not defined globally (only on equationally defined subsets);

Graphs:

Sorts: *edge* and *vertex*;
Operations: $s, t : edge \to vertex$ ($s =$source and $t =$target);
One partial operation $\sigma : edge \times edge \to edge$ s.t. $\sigma(x, y)$ is defined iff $s(x) = s(y)$ and $t(x) = t(y)$.
Axioms: $\sigma(x, y) = x$, $\sigma(x, y) = y$

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Theories in Logic

Regular Theories:

- Allow existential quantification over the usual equations (i.e. we can ask some maps to be surjective).

Von Neumann regular rings:

Models R of the regular theory with axioms

- associative rings with unit;
- $\forall a \, \exists b \; a = aba$; i.e.

$$\mathbb{Z}[x] \xrightarrow{f} A := \mathbb{Z}\langle x, y\rangle / x = xyx$$

$$\forall\, a : \mathbb{Z}[x] \to R, \quad \exists\, (a, b) : A \to R$$

i.e. the map $\mathrm{Rng}(A, R) \xrightarrow{- \circ f} \mathrm{Rng}(\mathbb{Z}[x], R)$ is surjective.

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Theories in Category Theory

Categorically speaking, we could think of a Theory as a category $\mathcal{C}$ with some structure, and of a model of $\mathcal{C}$ as a functor $F : \mathcal{C} \to \mathbf{Set}$ which preserves that structure.

Examples

1. Algebraic Theories: categories with finite products; their models are finite product preserving functors [Lawvere,63].
2. Essentially Algebraic Theories: categories with finite limits; lex functors are their models [Freyd,72].
3. Regular Theories: regular categories; their models are regular functors [Makkai-Reyes,77].

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Gabriel-Ulmer Duality

- The two notions of theory, categorical and logical, can be recovered from each other: given a logical theory, produce a category with the relevant structure for which models of the theory correspond to functors to **Set** preserving this structure, and vice versa.

For essentially algebraic theories there is a duality between theories and their models:

Theorem (Gabriel-Ulmer)

The following is a biequivalence of 2-categories:

$$\mathrm{Lfp}(-,\mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-,\mathbf{Set}).$$

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Regular and Exact Categories

Regular Categories: finitely complete ones with coequalizers of kernel pairs, for which regular epimorphisms are pullback stable.

Theorem (Barr's Embedding)

Let $\mathcal{C}$ be a small regular category; then the evaluation functor $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is fully faithful and regular.*

Exact Categories: regular ones with effective equivalence relations.

Theorem (Makkai's Image Theorem)

Let $\mathcal{C}$ be a small exact category. The essential image of the embedding $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is given by those functors which preserve filtered colimits and small products.*

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Duality for Exact Categories

- On one side of the duality there is the 2-category **Ex** of exact categories, regular functors, and natural transformations.
- On the other side is a 2-category **Def** whose objects are called definable categories and correspond to models of regular theories.

Theorem (Prest-Rajani/Kuber-Rosický)

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Enriched Categories

- Replace *Set* with a symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$.
- A $\mathcal{V}$-enriched category $\mathcal{B}$ is given by:
  1. a collection of objects $X, Y, Z, \ldots$;
  2. a morphism object $\mathcal{B}(X, Y)$ in $\mathcal{V}$, for each X, Y;
  3. composition maps $\mathcal{B}(Y, Z) \otimes \mathcal{B}(X, Y) \to \mathcal{B}(X, Z)$ in $\mathcal{V}$;
  4. identities $I \to \mathcal{B}(X, X)$, etc. (coherence axioms).
- A functor $F : \mathcal{B} \to \mathcal{C}$ between $\mathcal{V}$-categories is determined by:
  1. an object FX of $\mathcal{C}$ for each X in $\mathcal{B}$;
  2. a morphism $F_{XY} : \mathcal{B}(X, Y) \to \mathcal{C}(FX, FY)$ in $\mathcal{V}$, for each X, Y, satisfying some axioms.

Examples:

- $\mathcal{V} = Ab$, R-Mod for a commutative ring R;
- $\mathcal{V} = Cat, sSet, Ban, \ldots$

Varieties as Bases for Enrichment

- Our bases for enrichment will generally be models of (unsorted) algebraic theories, this are called finitary varieties.
- Categories of the form $\mathrm{FP}(\mathcal{C}, \mathbf{Set})$, consisting of finite product preserving functors for some small category $\mathcal{C}$ with finite products (and splitting idempotents).
- Equivalently a finitary variety can be described as an exact finitary quasivariety.

We consider the monoidal closed structures on $\mathrm{FP}(\mathcal{C}, \mathbf{Set})$ for which the tensor product restricts along the Yoneda embedding

$$Y : \mathcal{C}^{op} \hookrightarrow \mathrm{FP}(\mathcal{C}, \mathbf{Set})$$

giving a symmetric monoidal structure $(\mathcal{C}, \otimes, I)$.

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General Bases for Enrichment

Let $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ be a symmetric monoidal closed category.

Recall: An object A of $\mathcal{V}_0$ is called finite projective if the hom-functor $\mathcal{V}_0(A, -) : \mathcal{V}_0 \to \mathbf{Set}$ preserves filtered colimits and regular epimorphisms; denote by $(\mathcal{V}_0)_{pf}$ the full subcategory of finite projective objects.

Definition

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a symmetric monoidal closed category. We say that $\mathcal{V}$ is a symmetric monoidal finitary quasivariety if:

1. $\mathcal{V}_0$ is cocomplete with strong generator $\mathcal{P} \subseteq (\mathcal{V}_0)_{pf}$ (i.e. is a finitary quasivariety);
2. $I \in (\mathcal{V}_0)_f$;
3. if $P, Q \in \mathcal{P}$ then $P \otimes Q \in (\mathcal{V}_0)_{pf}$.

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Examples

1. **Set**, **Ab**, R-**Mod** and **GR**-R-**Mod**, for each commutative ring R, with the usual tensor product;
2. $[\mathcal{C}^{op}, \mathbf{Set}]$, for any category $\mathcal{C}$ with finite products, equipped with the cartesian product;
3. pointed sets $\mathbf{Set}_*$ with the smash product;
4. G-sets $\mathbf{Set}^G$ for a finite group G with the cartesian product;
5. directed graphs **Gra** with the cartesian product;
6. $\mathbf{Ch}(\mathcal{A})$ for each abelian and symmetric monoidal finitary quasivariety $\mathcal{A}$, with the tensor product inherited from $\mathcal{A}$;
7. torsion free abelian groups $\mathbf{Ab}_{tf}$ with the usual tensor product;
8. binary relations **BRel** with the cartesian product;

Regular $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-category $\mathcal{C}$ is called regular if:

- it has all finite weighted limits and coequalizers of kernel pairs;
- regular epimorphisms are stable under pullback and closed under powers by elements of $\mathcal{P} \subseteq (\mathcal{V}_0)_{pf}$.

$F : \mathcal{C} \to \mathcal{D}$ between regular $\mathcal{V}$-categories is called regular if it preserves finite weighted limits and regular epimorphisms.

- $\mathcal{V}$ itself is regular as a $\mathcal{V}$-category;
- if $\mathcal{C}$ is regular as a $\mathcal{V}$-category then $\mathcal{C}_0$ is a regular category;

Theorem (Barr's Embedding)

Let $\mathcal{C}$ be a small regular $\mathcal{V}$-category; then the evaluation functor $\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V}]$ is fully faithful and regular.

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Exact $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-category $\mathcal{B}$ is called exact if it is regular and in addition the ordinary category $\mathcal{B}_0$ is exact in the usual sense.

- Taking $\mathcal{V} = \mathbf{Set}$ or $\mathcal{V} = \mathbf{Ab}$ this notion coincides with the ordinary one of exact or abelian category.
- If $\mathcal{V}$ is a symmetric monoidal finitary variety, $\mathcal{V}$ is exact as a $\mathcal{V}$-category.

Theorem (Makkai's Image Theorem)

For any small exact $\mathcal{V}$-category $\mathcal{B}$; the essential image of $\mathrm{ev}_{\mathcal{B}} : \mathcal{B} \longrightarrow [\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V}]$ is given by those functors which preserve small products, filtered colimits and projective powers.

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Duality for Enriched Exact Categories

Assume $\mathcal{V}$ to be a symmetric monoidal finitary variety, then

- call a $\mathcal{V}$-category $\mathcal{D}$ definable if it is equivalent to $\mathrm{Reg}(\mathcal{C}, \mathcal{V})$ for a small regular $\mathcal{V}$-category $\mathcal{C}$;
- for each definable $\mathcal{D}$, the $\mathcal{V}$-category $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ is small and exact.

This and Makkai's Image Theorem imply:

Theorem

Let $\mathcal{V}$ be a symmetric monoidal finitary variety. Then the 2-adjunction

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Thank You

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