Accessible Categories with limits of some class

0) Introduction

Let $\mathcal{A}$ be an accessible category:

i) (Gabriel-Ulmer) $\mathcal{A}$ is complete $\Leftrightarrow$ $\mathcal{A}$ is cocomplete ($\emptyset^*\mathcal{A}$ is cocomplete)

ii) (Diers) $\mathcal{A}$ with connected limits $\Leftrightarrow$ $\mathcal{A}$ is multicocomplete ($\mathrm{Fam}^*\mathcal{A}$ is coc.)

iii) (Adamek-Rosicky) $\mathcal{A}$ with small products $\Leftrightarrow$ $\mathcal{A}$ is weakly cocomplete ?

iv) (Lamarche) $\mathcal{A}$ with wide pullbacks $\Leftrightarrow$ $\mathcal{A}$ is poly-cocomplete ?

v) (Hève-T. Lair) $\mathcal{A}$ has $\emptyset$-limits $\Leftrightarrow$ $\mathcal{A}$ is virtually cocomplete (pre-cocomplete) ($\mathrm{Lim}\,\mathcal{A}$ cocomplete)

$\mathcal{A}$ with Ψ-limits $\Leftrightarrow$ $\underset{\substack{\parallel \\ \Psi\text{-}\mathrm{Cont}(\mathcal{A},\underline{Set})^{op} \cap \underset{\substack{\parallel \\ [\mathcal{A},\underline{Set}]^{op}}}{\mathrm{Lim}\,\mathcal{A}}}}{D^*\mathcal{A}}$ is cocomplete $\mathrm{Lim}\,\mathcal{A} = \mathcal{P}(\mathcal{A}^{op})^{op}$

1) Locally multi-presentable (l.m.p) categories

Def: $\mathcal{A}$ is l.m.p if it is accessible and multi-cocomplete

Def: Let $H: \mathcal{C} \to \mathcal{A}$ be a diagram; a multi-colimit of H in $\mathcal{A}$ is $(A_i)_{i\in I}$ t.w. $(H \xrightarrow{c_i} \Delta A_i)_i$ s.t. $\forall\ H \xrightarrow{c} \Delta A$ $\exists!\, i$ and a unique factn. of c through c_i as $A_i \to A$.

Example: • Fields is l.m.p

↳ multi initial object: $\mathbb{Q} \cup \{\mathbb{Z}/p\mathbb{Z}\}_{p \text{ prime}}$

$\forall\ \mathbb{F}$ $\mathbb{Z}/_{\mathrm{char}(\mathbb{F})}\mathbb{Z} \xrightarrow{!} \mathbb{F}$ $\mathrm{char}(\mathbb{F}) \neq 0$

$\mathbb{Q} \xrightarrow{!} \mathbb{F}$

• Linearly ordered sets, Hilbert Spaces, local rings ..

Remark: If $(A_i)_i$ is a multi-colimit of $H: \mathcal{C} \to \mathcal{A}$, and $V: \mathcal{A} \hookrightarrow \underset{\substack{\text{free compl.} \\ \text{und. products} \\ \parallel}}{\mathrm{Fam}^*\mathcal{A}}$

$\coprod_i VA_i$ is colimit of VH. Vice versa if $X = \mathrm{colim}\, VH$

$\Rightarrow X = \coprod_i VA_i \rightsquigarrow (A_i)$ is a multi-colimit of H

Prop: $\mathcal{A}$ is l.m.p $\Leftrightarrow$ it is accessible with connected limits

• connected limits commute in $\underline{Set}$ with coproducts

2) Locally poly-presentable categories (l.p.p.)

Def: $\mathcal{A}$ is l.p.p. if it is accessible and <u>poly-cocomplete</u>

Def: Let $H: \mathcal{C} \to \mathcal{A}$ be a diagram; a poly-colimit of H in $\mathcal{A}$ is a family $(A_i)_i$ t.w. $(H \xrightarrow{c_i} \Delta A_i)_i$ such that $\forall\ H \xrightarrow{c} \Delta A$ $\exists!\, i$ for which c factors through c_i as $A_i \xrightarrow{f} A$, and such f is unique up to a unique automorphism of A_i: $f', f : A_i \to A$ which give $c \Rightarrow \exists!\ g: A_i \xrightarrow{\cong} A_i$ s.t. $f = f'g$.

Example: • Algebraically closed fields is l.p.p.

poly-initial object: $\overline{\mathbb{Q}} \cup \{\overline{\mathbb{Z}/p\mathbb{Z}}\}_{p \text{ prime}}$

$$\left(\begin{tikzcd} & \overline{\mathbb{Q}} \arrow[dr, "g"] & \\ \overline{\mathbb{Q}} \arrow[ur, dashed, "!_{\text{id}}"] \arrow[rr, "f"'] & & \mathbb{C} \end{tikzcd}\right)$$
$\cancel{\ast}$

• categories of Embedding - Coquand

Prop: $\mathcal{A}$ is l.p.p $\Leftrightarrow$ $\mathcal{A}$ is accessible with wide pullbacks

Hu-Tholen: free groupoid actions in Set

• colimits of those commute in $\underline{Set}$ with wide pullbacks

3) Weakly locally presentable categories (w.l.p)

Def: $\mathcal{A}$ is w.l.p. if it is accessible and weakly-cocomplete

Def: Let $H: \mathcal{C} \to \mathcal{A}$ be a diagram; a weak colimit of H in $\mathcal{A}$ is an object A t.w. $H \xrightarrow{c} \Delta A$ s.t. every $H \xrightarrow{d} \Delta B$ factors through it as $A \to B$. (A is not unique itself)

Examples: Definable categories are w.l.p. (injectivity classes)

• divisible modules over a ring ($\forall x \in M\ \forall r \in R\ \exists y \in M$ s.t. $ry = x$)

• torsion free modules over a ring.

Remark: Given $\mathcal{A}$, $(W^*\mathcal{A} = W(\mathcal{A}^{op})^{op})$

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "Y"] \arrow[dr, hook] & {[\mathcal{A}^{op}, \underline{Set}]} \\ & W\mathcal{A} \arrow[u, hook] \end{tikzcd}$$
($\exists X$ is s.t. $\exists\, YA \xrightarrow{\text{(r) epi}} X$)

$\mathcal{A}$ is weakly cocomplete $\Leftrightarrow$ $W^*\mathcal{A}$ has colimits of reps.

• $W^*\mathcal{A} \not\subseteq \mathrm{Lim}\,\mathcal{A}$

• we want to identify a class of colimits which commute with products in $\underline{Set}$

(Adamek, $\mathcal{C}$-colimits commute with products in $\underline{Set}$ $\Rightarrow$ they are absolute)

4) The general Setting ($\mathcal{V} = \underline{Set}$)

• From now on $\underline{\Psi}$ is a <u>class of indexing categories</u>

- {small connected categories} $\rightsquigarrow$ conn. limits
- { " discrete cats} $\rightsquigarrow$ products
- { $\cdot \to \cdot \leftarrow \cdot$ ...} $\rightsquigarrow$ wide pullbacks
- {small cats} $\rightsquigarrow$ all limits

[Def: A class of diagrams $\mathbb{D}$ is the data of a full subcategory $\mathbb{D}_{\mathcal{C}} \subseteq [\mathcal{C}, \underline{Set}]$ for each small cat $\mathcal{C}$.

($\mathbb{D}_{\mathcal{C}} \hookrightarrow [\mathcal{C}, \underline{Set}]$ accessible, acc. embedded + ...)

Examples: 1) Any Ψ can be seen as a class of diagrams

$$\begin{cases} \mathbb{D}_{\mathcal{C}} = \emptyset & \mathcal{C} \notin \Psi \\ \mathbb{D}_{\mathcal{C}} = [\mathcal{C}, \underline{Set}] & \mathcal{C} \in \Psi \end{cases}$$

2) $\mathcal{F}$ of free groupoid diagrams in $\underline{Set}$

$$\begin{cases} \mathcal{F}_{\mathcal{G}} = \emptyset & \mathcal{G} \text{ not a groupoid} \\ \mathcal{F}_{\mathcal{G}} \subseteq [\mathcal{G}, \underline{Set}] & \mathcal{G} \text{ is a groupoid} \end{cases}$$
$F: \mathcal{G} \to \underline{Set}$, $\mathcal{G} = \coprod_i G_i$ $F = (F_i : G_i \to \underline{Set})$

$F \in \mathcal{F}_{\mathcal{G}} \Leftrightarrow$ each F_i is a free action ($F_i(g)$ has fixed points $\Leftrightarrow g = \mathrm{id}: G_i \to G_i$)

3) $\mathcal{R}$ of pseudo equivalence relations in $\underline{Set}$

$$\begin{cases} \mathcal{R}_{\mathcal{C}} = \emptyset & \mathcal{C} \neq \{\cdot \rightrightarrows \cdot\} \\ \mathcal{R}_{\{\cdot \rightrightarrows \cdot\}} \subseteq [\cdot \rightrightarrows \cdot, \underline{Set}] & \text{is given by the p. equiv. relations in } \underline{Set} \end{cases}$$

$X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is a p. equiv. rel $\Leftrightarrow$ $\left(\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\text{epi}"'] & Y \\ E \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}\right)$ ← kern. pairs

Def: Given $\mathbb{D}$ a class of diagrams, and Ψ a class of index. cats. We say that $\mathbb{D}$-colimits commute with Ψ-limits in $\underline{Set}$ if: $\forall\ \mathcal{C}$ the colimit functor $\mathrm{colim}: [\mathcal{C}, \underline{Set}] \to \underline{Set}$ preserves Ψ-limits of diagrams landing in $\mathbb{D}_{\mathcal{C}} \overset{Z}{\hookrightarrow} [\mathcal{C}, \underline{Set}]$

$\forall\ \mathcal{B} \in \Psi$ and $H: \mathcal{B} \to \mathbb{D}_{\mathcal{C}}$ $\mathrm{colim}(\lim ZH) \cong \lim(\mathrm{colim}\, ZH)$

Examples: i) $\mathcal{F}$-colimits commute with wide pullbacks in $\underline{Set}$

ii) $\mathcal{R}$-colimits commute with products in $\underline{Set}$

$$\left(X_i \rightrightarrows Y_i \to Z_i \rightsquigarrow \prod X_i \rightrightarrows \prod Y_i \to \prod Z_i \right)$$
(k.p.)

Def: We say that $\mathbb{D}$ is a companion for Ψ if:

i) $\mathbb{D}$-colimits commute with Ψ-limits in $\underline{Set}$;

ii) $\forall\ \mathcal{A}$ Ψ-complete and virtually cocomplete ($\mathrm{Lim}\,\mathcal{A}$ is cocomplete) every <u>small</u> functor $F: \mathcal{A} \to \underline{Set}$, which is Ψ-continuous, is a $\mathbb{D}$-colimit of representables: $Y: \underline{\mathcal{A}^{op}} \to [\mathcal{A}, \underline{Set}]$,

$\exists\ H: \mathcal{C} \to \mathcal{A}^{op}$ s.t. $\mathcal{A}(H\_, A) \in \mathbb{D}_{\mathcal{C}}$ $\forall A$ and $\mathrm{colim}\, YH \cong F$

• Notation: $\mathbb{D}(\mathcal{A}^{op})$ the full subcat of $[\mathcal{A}, \underline{Set}]$ given by the representables and $\mathbb{D}$-colimits of those.

• If $\mathbb{D}$ is a companion for Ψ, and $\mathcal{A}$ is Ψ-complete and virt. cocomplete

$$\mathbb{D}(\mathcal{A}^{op}) = \Psi\text{-}\mathrm{Cont}(\mathcal{A}, \underline{Set}) \cap \underbrace{\mathcal{P}(\mathcal{A}^{op})}_{\text{small colimits of reps.}}$$
$\subseteq$ is given by (i)

$\supseteq$ is given by (ii)

Examples:

1) $\mathcal{F}$ is a companion for wide pullbacks
2) $\mathcal{R}$ " " for products
3) $\mathbb{D} = \emptyset$ " " for $\mathcal{P} = \{\text{all small cats}\}$
4) $\mathbb{D} = \{\text{discrete cats}\}$ (coproducts) is a comp. for $\Psi = \{\text{connected cats}\}$
5) $\mathbb{D} = \mathcal{P}$ is a companion for $\Psi = \emptyset$
6) $\mathbb{D} = \{\text{filtered cats}\}$ is a comp. $\Psi = \{\text{finite cats}\}$
7) $\mathbb{D} = \{\text{sifted cats}\}$ " $\Psi = \{\text{finite discrete cats}\}$ (finite prod)

Theorem: Let $\mathbb{D}$ be a companion for Ψ and $\mathcal{A}$ be a category; ($\mathbb{D}^*\mathcal{A} = \mathbb{D}(\mathcal{A}^{op})^{op}$)

TFAE:

1) $\mathcal{A}$ is accessible with Ψ-limits;
2) $\mathcal{A}$ is accessible and $\mathbb{D}^*\mathcal{A}$ is cocomplete;
3) $\mathcal{A}$ is accessible and $\mathbb{D}^*\mathcal{A}$ has colimits of representables;
4) $\mathcal{A}$ is accessibly embedded and $\mathbb{D}^*$-reflective in $[\mathcal{C}, \underline{Set}]$ for some $\mathcal{C}$

$$\begin{tikzcd} \mathbb{D}^*\mathcal{A} & \\ \mathcal{A} \arrow[u, hook, "V"] \arrow[r, hook, "J"'] & {[\mathcal{C}, \underline{Set}]} \arrow[ul, dashed, bend right, "L"'] \end{tikzcd}$$
$L \dashv_V J$

($\mathbb{D}^*\mathcal{A}(LX, VA) \cong [\mathcal{C}, \underline{Set}](X, JA)$ nat. in X, A)

(eq. $L \dashv \mathrm{Ran}_V J$)

e.g. $\begin{cases} \Psi = \{\text{wide pullbacks}\} \\ \mathcal{A} \text{ is acc. [with wide pullbacks} \Leftrightarrow \mathcal{F}^*\mathcal{A} \text{ has colimits of reps.]} \end{cases}$

$\Updownarrow$

$\mathcal{A}$ is poly-cocomplete

End

($\forall\ \varphi: \mathcal{C}^{op} \to \mathcal{V}$ $\underline{\mathbb{D}_{\varphi} \subseteq [\mathcal{C}, \mathcal{V}]}$) { Hu-Tholen, Quasi-coproducts...

$\begin{cases} \mathbb{D}_{\mathcal{C}} \hookrightarrow [\mathcal{C}, \underline{Set}] \text{ is accessible and acc. embed.} \\ \parallel \\ \mathrm{Mod}(S, \mathbb{L}, \mathbb{C}) \end{cases}$

• Lamarche - PhD then
• Paul Taylor: Locally poly pres. categories

($\mathbb{D}_{\mathcal{C}} \subseteq [\mathcal{C}, \underline{Set}]$)

$\rightarrow$ $\forall\ \mathcal{A}$ $\mathbb{D}_{\mathcal{A}} \subseteq [\mathcal{C}, \mathcal{A}]$

$\mathcal{A} \overset{V}{\hookrightarrow} \mathcal{F}\mathcal{A}$ $\begin{cases} H: \mathcal{C} \to \mathcal{F}\mathcal{A} \\ \mathcal{F}\mathcal{A}(VA, H\_) \in \mathcal{F} \end{cases}$

$\Downarrow$

Hu Tholen $\begin{cases} H: \mathcal{C} \to \mathcal{F}\mathcal{A} \\ \text{s.t. } \forall X \in \mathcal{F}\mathcal{A} \text{ not initial} \quad \mathcal{F}\mathcal{A}(X, H\_) \in \mathcal{F} \end{cases}$

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