On Small Accessible $\mathcal{V}$-categories and continuity of accessible $\mathcal{V}$-functors.

$\mathcal{V} = \underline{Set}$ (Makkai-Paré)

A small category $\mathcal{C}$ is accessible
$\Leftrightarrow$ idempotents split in $\mathcal{C}$
$\Leftrightarrow$ $\mathcal{C}$ is Cauchy complete

$\mathcal{V}$ (Today)

A small $\mathcal{V}$-category $\mathcal{C}$ is accessible
$\Leftrightarrow$ it is Cauchy Complete

I) $\mathcal{V} = Set$.

* Every accessible category has splittings of idempotents
* Let $\mathcal{C}$ be small and have splittings of idempotents.

Take β s.t. $\#|\mathcal{C}| < \beta$, take $\gamma > \beta$

γ-filtered colimits in $\mathcal{C}$ = splitting of idem.

every object of $\mathcal{C}$ is γ-presentable.
So $\mathcal{C}$ is γ-accessible.

II) Setting:

$\mathcal{V}$ is symm. monoidal closed and locally presentable.

fix α_0 s.t. $\mathcal{V}$ is loc. α pres as a closed category.

[From now on every cardinal is $\geq \alpha_0$.

Def: Let $\mathcal{A}$ be a $\mathcal{V}$-category.

- $\mathcal{A}$ is <u>conically accessible</u> if it is the free cocompletion of a small $\mathcal{V}$-category under α-filtered colimits, for some α.

- $\mathcal{A}$ is <u>accessible</u> if $\mathcal{A} \simeq \alpha\text{-}\mathrm{Flat}(\mathcal{C},\mathcal{V})$, for a small $\mathcal{C}$ and some α.

We work with this.

A small $\mathcal{V}$-category is conically accessible $\Leftrightarrow$ it has splittings of idempotents.

III) Flatness

Def: Let $\varphi : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$-functor, we say that φ is
(Kelly). α-small if

i) $\mathcal{C}$ is α-small : $|\mathrm{Ob}(\mathcal{C})| < \alpha$, $\mathcal{C}(X,Y) \in \mathcal{V}_\alpha$

ii) $\varphi(X) \in \mathcal{V}_\alpha \quad \forall\, X \in \mathcal{C}$.

- α-small limits are those weighted by α-small functors.

Def: We say that $\varphi : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ is α-flat if $\mathrm{Lan}_Y \varphi$ is α-continuous.

$$\begin{tikzcd}
{[\mathcal{C}^{op},\mathcal{V}]} \arrow[dr, "\mathrm{Lan}_Y \varphi"] & \\
\mathcal{C}^{op} \arrow[u, "Y"] \arrow[r, "\varphi"'] & \mathcal{V}
\end{tikzcd}$$

But $\mathrm{Lan}_y \varphi \cong \varphi * -$.

Equivalently: φ colimits commute in $\mathcal{V}$ with α-small limits.

Prop: Let $\varphi : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$ functor; TFAE:

i) φ is α-flat

ii) $\varphi * -$ preserves α-small limits of representables

If $\mathcal{C}^{op}$ is α-complete, moreover:

iii) φ is α-continuous.

<u>Lemma</u>: Given $\varphi : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ and $J : \mathcal{C} \longrightarrow \mathcal{D}$.

i) φ is α-flat $\Rightarrow$ $\mathrm{Lan}_{J^{op}} \varphi$ is α-flat.

ii) J is f.f and $\mathrm{Lan}_{J^{op}} \varphi$ is α-flat $\Rightarrow$ φ is α-flat as well.

<u>Def</u>: $\varphi : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ is Cauchy if it is α-flat $\forall \alpha$.

i.e. $\varphi * - : [\mathcal{C}, \mathcal{V}] \longrightarrow \mathcal{V}$ is continuous.

$\Longleftrightarrow$ φ-colimits are preserved by any $\mathcal{V}$-functor.

(φ-colimits are absolute).

We say that $\mathcal{C}$ is Cauchy complete if it has all absolute colimits, i.e all colimits weighted by Cauchy functors.

Fact: $\mathcal{C}$ is Cauchy complete $\Leftrightarrow$ $\mathcal{C} \simeq \mathrm{Cauchy}(\mathcal{C}^{op}, \mathcal{V}) \subseteq [\mathcal{C}^{op}, \mathcal{V}]$

IV) The Result.

★ If $\mathcal{C}$ is (small and) accessible $\Rightarrow$ $\mathcal{C}$ is Cauchy complete

↳ Because Cauchy-colimits are α-flat (for any α)

★ Let $\mathcal{C}$ be Cauchy complete and Small, we want to find γ s.t.

$$\mathcal{C} \simeq \mathrm{Cauchy}(\mathcal{C}^{op}, \mathcal{V}) \underset{\supseteq \text{ need to prove}}{\overset{\subseteq \text{ OK}}{=}} \gamma\text{-Flat}(\mathcal{C}^{op}, \mathcal{V}).$$

then $\mathcal{C}$ is γ-accessible (by definition).

<u>Proposition</u>: If $\varphi : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ is α-small and α-flat then it is Cauchy.

↳ Proof: Let $\mathcal{D} = \Phi^{\dagger}_{\alpha}(\mathcal{C}^{op})$ ← free completion under α-small limits.

$J : \mathcal{C}^{op} \hookrightarrow \mathcal{D}$ inclusion.

We can take $X := \{\varphi, J\} \in \mathcal{D}$, then

$$\mathrm{Lan}_J \varphi \cong \mathcal{D}(X, -) \qquad (\text{use that } \varphi \text{ is } \alpha\text{-flat})$$

But $\mathcal{D}(X,-)$ is Cauchy, then $\mathrm{Lan}_J \varphi$ is Cauchy $\Rightarrow$ φ is Cauchy by the Lemma.

$\square$

Now: Given $\mathcal{C}$, it's enough to find γ such that every γ-flat $\varphi : \mathcal{C}^{op} \rightarrow \mathcal{V}$ is γ-small.

Easy for $\mathcal{V} = \underline{Set}$: (Makkai - Paré)

Lemma: Let $\mathcal{C}$ be an small category. Take β s.t. $\mathcal{C}$ is β-small, and $\gamma > \beta$. then any γ-flat $\varphi : \mathcal{C}^{op} \to \underline{Set}$ is β-small, and in particular γ-small.

↳ Proof: Assume $\exists\, X \in \mathcal{C}$ s.t. $\varphi(X)$ has β distinct elements $\{x_i\}_{i \in I}$, $|I| = \beta$.

$(x_i, X) \in \mathrm{El}(\varphi) \leftarrow \gamma$-filtered

$\exists\ (y, Y) \in \mathrm{El}(\varphi)$ t.w $\quad (x_i, X) \xrightarrow{f_i} (y, Y)$ in $\mathrm{El}(\varphi)$.

$$\varphi(f_i)(y) = x_i$$

$\mathcal{C}$ is β-small, the $\{f_i\}_i < \beta$. This is a contradiction because the x_i's are all distinct. □

Prop: Let $\mathcal{C}$ be a small $\mathcal{V}$-category. then there is γ such that every γ-flat $\varphi : \mathcal{C}^{op} \to \mathcal{V}$ is γ-small. In particular every γ-flat φ is Cauchy.

↳ (Sketch) (I) $\mathcal{C}$ α-complete for some α.

First take $\beta > \alpha$.

i) $\mathcal{C}$ is β-small as a $\mathcal{V}$-category.

ii) For any $X \in \mathcal{V}$

$$X \in \mathcal{V}_\beta \iff \mathcal{V}_0(Y, X) \in \mathrm{Set}_\beta \ \forall\ Y \in \mathcal{V}_\alpha$$

then take $\gamma > \beta$.

(II) Take any Cauchy complete $\mathcal{C}$ and use (I)

Theorem: A small $\mathcal{V}$-category $\mathcal{C}$ is accessible $\Leftrightarrow$ it is Cauchy Complete. $\square$

Prop: Let K be locally α-presentable. Then there exist γ such that any α-accessible (preserves α-filt colimits) $\mathcal{V}$-functor $F : K \longrightarrow \mathcal{L}$ ($\mathcal{L}$ α-accessible $\mathcal{V}$-category) is continuous if and only if it preserves γ-small limits.

↳ Proof: take γ as in the Proposition for $\mathcal{C} = K_\alpha$.

Wlog $\mathcal{L} = \mathcal{V}$.

$$\begin{tikzcd}
{[K_\alpha^{op}, \mathcal{V}]} \arrow[ddr, bend left, "FJ * -"] & \\
\alpha\text{-Cont}(K_\alpha^{op}, \mathcal{V}) = K \arrow[u, hook, "\text{continuous}"] \arrow[dr, "F = \mathrm{Lan}_J(FJ)"] \arrow[r, phantom, "\circlearrowleft"] & \\
K_\alpha \arrow[u, hook, "J"] \arrow[r, "FJ"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{V}
\end{tikzcd}$$

- F γ-continuous

$\Downarrow$

- $FJ * -$ preserves γ-small limits of representables

$\Downarrow$

- FJ is γ-flat

$\Downarrow$

- FJ is Cauchy

$\Downarrow$

- $FJ * -$ is continuous

$\Downarrow$

- F is continuous. $\square$

Corollary: $X \in \mathcal{V}$ is dualizable $\Leftrightarrow$ $X \otimes -$ is γ-continuous.