Some facts about Sound classes of weights

Enriched version of "Adamek, Borceux, Lack, Rosicky"

Setting: Sym. mon. closed category $(\mathcal{V}_0, \otimes, I)$ which is locally bounded (eg. loc. pres, Top, ...)

Def: Let Φ be a class of weights $(\Phi = \{M : \mathcal{C}^{op} \to \mathcal{V}\})$

Let $M : \mathcal{C}^{op} \to \mathcal{V}$ be a $\mathcal{V}$-functor, we say that M is Φ-flat is M-colimits commute in $\mathcal{V}$ with Φ-limits

$$M * - : [\mathcal{C}, \mathcal{V}] \longrightarrow \mathcal{V} \quad \text{is } \Phi\text{-continuous.}$$

Def: i) We say that Φ is sound if $\forall\, M : \mathcal{C}^{op} \to \mathcal{V}$ for which $M * -$ preserves Φ-limits of representables, then M is Φ-flat.

ii) We say that Φ is weakly sound if $\forall\, M : \mathcal{C}^{op} \to \mathcal{V}$ which is Φ-continous ($\mathcal{C}^{op}$ is Φ-complete), then M is Φ-flat.

Ex: $\mathcal{V} = \mathbf{Set}$, $\Phi = \{$finite categories$\}$

$\forall\, \mathcal{C}$, $\Delta 1 : \mathcal{C}^{op} \to \mathbf{Set}$, then $\Delta 1$ is Φ-flat iff $\mathcal{C}$ is filtered

$M : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ is Φ-flat $\iff$ $\mathrm{El}(M)$ is filtered

$\iff$ M is a filt. colimit of rep. $\square$

Easy: Φ sound $\Rightarrow$ Φ weakly sound

$(\Leftarrow)$ Is not true in general eg. $\underline{PT} = \{ \underbrace{\cdot \to \dot{\downarrow}, \emptyset}\}$ $\mathcal{V} = \mathrm{Set}$

Def: For any $\mathcal{C}$ let $\Phi\mathcal{C}$ be the free cocompletion of $\mathcal{C}$ under Φ-colimits. $(\mathcal{C} \hookrightarrow \Phi\mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathcal{V}])$

Say that Φ is presaturated if $\forall$ $\mathcal{C}$, any $X \in \Phi\mathcal{C}$ is a Φ-colimit of elements in $\mathcal{C}$.

Note: $\underline{PT}$ is not presaturated, while $\underline{\mathrm{Fin}} = \{\text{finite categories}\}$ is.

Prop: Let Φ be presaturated, then Φ is sound $\iff$ it is weakly sound.

Examples:

i) $\Phi = \emptyset$ (any $\mathcal{V}$)

ii) $\mathcal{V} = \mathrm{Set}$ $\Phi = \mathrm{Fin}$, $\Phi_\alpha = \{\alpha\text{-small cat}\}$

iii) $\mathcal{V} = \mathrm{Set}$ $\Phi = \{\emptyset\}$

iv) $\mathcal{V} = \mathrm{Set}$ $\Phi = \{\text{finite discrete cats}\}$

v) $\mathcal{V} = \mathrm{Set}$ $\Phi = \{\text{finite connected categories}\}$

vi) $\mathcal{V}$ l.f.p. as closed cat $\Phi = \{\text{finite weights}\}$

$\mathcal{V} = \mathrm{Cat}$ $\Phi = \langle \text{fin. connected cats, powers by fin. conn. cats} \rangle$

vi) $\mathcal{V}$ (locally bounded) and cartesian closed, $\Phi = \{$ finite discret categories $\}$ is (by eg. Kelly-Lack) ($\mathcal{V} =$ CGTop, QTop)

Now Fix $\mathcal{V}$ locally bounded and a small weakly sound class Φ

Def: Let K be a cocomplete $\mathcal{V}$-category, $X \in K$ is Φ-presentable if $K(X,-) : K \longrightarrow \mathcal{V}$ preserves Φ-colimits, say $X \in K_\Phi$.
We say tha K is locally Φ-presentable if there is $\mathcal{C} \subseteq K_\Phi$ small and a strong generator for K ($K \underset{\text{conservative}}{\longrightarrow} [\mathcal{C}^{op}, \mathcal{V}]$)

[Scott Johnson: $\mathcal{V}$ is locally presentable then Cauchy completions are small]
Cauchy ↳ adding absolute colimits.

Prop: Let K be loc. Φ-presentable then K_Φ is the Cauchy completion of a small $\mathcal{V}$-cat $\mathcal{C} \subseteq K_\Phi$.

Theorem: Let K be a $\mathcal{V}$-category; TFAE:

1) K is loc. Φ-presentable.
2) $K \simeq \Phi\text{-Cont}(\mathcal{C}, \mathcal{V})$ for some small Φ-complete $\mathcal{C}$.
3) $K \simeq \mathrm{Mod}(S, \mathcal{V})$ where S is a Φ-limit sketch.
4) K is a Φ-orthogonality class in some $[\mathcal{C}, \mathcal{V}]$.
5) $K \underset{J}{\overset{L}{\rightleftarrows}} [\mathcal{C}, \mathcal{V}]$ J preserves Φ-flat colimits.

Note → $5 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ hold for any $\mathcal{V}$.

→ $4 \Rightarrow 5$ holds if $\mathcal{V}$ is loc. bounded (Kelly)

<u>Theorem:</u> there is a duality:

$$\begin{tikzcd}
\Phi\text{-Cont}(\mathcal{C},\mathcal{V}) & \mathcal{C} \arrow[l, mapsto] \\
L\Phi P \arrow[r, shift right] & \Phi\text{-Cont} \arrow[l, shift right] \\
K \arrow[r, mapsto] & K_{\Phi}^{op}
\end{tikzcd}$$

obj: K loc Φ-pres.

morph: $F : K \longrightarrow \mathcal{L}$ cont + Φ-flat colimit preserving

...

• $\mathcal{C}$ is Φ-complete and the Cauchy compl. of a small $\mathcal{V}$-cat.

• Φ-continuous functors.

...

• <u>Applications</u>

<u>Set</u>, Φ a small weakly sound class. ($\Phi = \{$finite cats$\}$)

→ $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is locally Φ-presentable as a closed category if

i) $\mathcal{V}_0$ is loc. Φ-presentable

ii) A, B Φ-presentable $\Rightarrow A \otimes B \in (\mathcal{V}_0)_\Phi$; $I \in (\mathcal{V}_0)_\Phi$

Eg: 1) $\Phi = \{$fin. cats$\}$ ⇝ loc. fin. pres. as a closed category (Kelly)

(Set, Ab, Cat, SSet, DGAb, ...)

2) $\Phi = \{$fin discr. categories$\}$ → $\mathcal{V}_0$ is a finitary variety + ...

(Set, Ab, GAb)

Prop: If $\mathcal{V}$ is locally Φ-presentable as a closed category then $\Phi_{\mathcal{V}} = \langle \Phi, \text{ powers by elements in } (\mathcal{V}_0)_\Phi \rangle$ is weakly sound (in the enriched sense).

Prop: Let Q be the class of Cauchy weights; then

$$Q \subseteq (\Phi_{\mathcal{V}} \cup S)^*.$$

(split coequalizers)

Cor: Cauchy completions are small if $\mathcal{V}_0$ is l.p. (Scott Johnson)

Let $\mathcal{V}$ be loc. presentable $\Rightarrow$ find α s.t. $\mathcal{V}$ is loc. α-presentable as a closed category.

$\Rightarrow \quad Q \subseteq \Phi_\alpha^*$ this is small

So Cauchy completions are small. ⌟

- Take $\mathcal{V} = Ab$, $\Phi = \langle \text{finite discr. cats} \rangle$

  so $\Phi_{Ab} = \langle \text{finite direct sums} \rangle$

  $\Rightarrow \quad Q \subseteq \Phi_{Ab} \cup S \subseteq Q$

  $\Rightarrow \quad Q = \Phi_{Ab} \cup S$ (as we knew)

- Take $\mathcal{V} = GAb$, $\Phi = \langle \text{finite discr. cat.} \rangle$

So $\quad \Phi_{GAb} = \langle$ finite direct sums, suspensions and desuspensions $\rangle$

$\Longrightarrow \quad Q \subseteq \Phi_{GAb} \cup S \subseteq Q$

$\Longrightarrow \quad Q = \langle$ finite direct sums, suspensions ~~and~~ desuspensions, split coeq. $\rangle$

(as in my paper with Branko and Ross)