Accessible Categories made Sound

[ABLR]: Adamek - Borceux - Lack - Rosicky

- $\mathcal{A}$ is accessible $\overset{\text{def}}{\iff}$ $\mathcal{A}$ is α-accessible for some regular cardinal α

- $\mathcal{A}$ is α-accessible if it is the free cocompletion of a small cat $\mathcal{C}$ under α-filtered colimits : $\mathcal{A} \simeq \alpha\text{-Ind}(\mathcal{C})$

  ($\mathcal{A} \supseteq \mathcal{A}_\alpha$ which generates under α-filtered colimits)
  $\mathcal{C} = \mathcal{A}_\alpha^{op}$

Why is this important?, and the theory works C^∞

$D_\alpha := \{\alpha\text{-small categories}\}$, $D_\omega = \{$ finite categories $\}$ $\longleftarrow$ $\left\{ \begin{array}{c} \text{shapes of} \\ \text{the limits} \\ \text{you want to} \\ \text{consider} \end{array} \right\}$

$\longrightarrow$ • α-filtered colimits are those that commute with α-small limits in $\mathbf{Set}$

$\longrightarrow$ (Soundness) ("good") $\mathcal{C}$ is α-cocomplete $\Rightarrow$ it is α-filtered

• The General setting:

Let $\mathbb{D}$ be a doctrine : a small set of categories

- a $\mathbb{D}$-limit in a $\mathcal{A}$ is the limit of $H : \mathcal{D} \longrightarrow \mathcal{A}$ where $\mathcal{D} \in \mathbb{D}$.

Def: Say that a small category $\mathcal{C}$ is $\mathbb{D}$-filtered if $\mathcal{C}$-colimits commute in $\underline{\mathrm{Set}}$ with $\mathbb{D}$-limits.

(def of [ABLR])

Def: $\mathbb{D}$ is <u>strongly sound</u>: if $\mathcal{C}$ is s.t. : $\forall$ $\underline{F : D^{op} \to \mathcal{C}}$, $\underline{D \in \mathbb{D}}$

the category of cocones of F is connected

obj : $\eta : F \longrightarrow \Delta A$

morph :

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\gamma"'] & \Delta A \arrow[d, "\Delta f"] \\ & \Delta B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

Then $\mathcal{C}$ is $\mathbb{D}$-filtered

Def: We say that $\mathbb{D}$ is sound if $\forall \mathcal{C}$: $\underline{\text{if } \mathcal{C} \text{ is } \mathbb{D}\text{-cocomplete}}$ $\Rightarrow$ $\underline{\mathcal{C} \text{ is } \mathbb{D}\text{-filtered.}}$

Strongly sound $\Rightarrow$ Sound (Prop 2.5 in [ABLR])

Examples: The following are sound doctrines $\qquad$ $\mathbb{D}$-filtered

i) $\underline{\mathbb{D}_\omega}$, $\mathbb{D}_\alpha = \{\alpha\text{-small cats}\}$ (α reg. card) $\longleftrightarrow$ α-filtered

ii) $\mathbb{D}_{pr} = \{\text{finite discrete categories}\}$ $\longleftrightarrow$ sifted cat

iii) $\mathbb{D}_{con} = \{\text{finite connected categories}\}$?

iv) $\mathbb{D}_{Term} = \{\emptyset\}$ $\longleftrightarrow$ (poss. infinite) connected cats

v) $\emptyset$ $\longleftrightarrow$ all cats

vi) $\underline{\mathbb{D}_{weird}} = \{\emptyset\} \cup \{\to \downarrow\}$ is $\underline{\text{sound but not strongly sound.}}$

[Conjecture: If $\mathbb{D}$ is "saturated" $\Rightarrow$ sound = strongly sound]

Def: Let $\mathbb{D}$ be a class of indexing categories; given $\mathcal{A}$ we denote by $\mathbb{D}(\mathcal{A})$ the free cocompletion of $\mathcal{A}$ under $\mathbb{D}$-colimits.

universal prop. $\dfrac{\mathcal{A} \xrightarrow{F} K \quad \text{↖ } \mathbb{D}\text{-cocomplete}}{\mathbb{D}\mathcal{A} \xrightarrow[\mathrm{Lan}_J F]{} K \quad \mathbb{D}\text{-cocontinuous}}$

$\mathcal{A} \overset{J}{\hookrightarrow} \underline{\mathbb{D}\mathcal{A}} \subseteq [\mathcal{A}^{op}, Set]$

↳ the $\underline{\text{closure}}$ of the reps. in $[\mathcal{A}^{op}, Set]$ under $\mathbb{D}$-colimits.

Given a doctrine $\mathbb{D}$, can consider

- $\mathbb{D}(\mathcal{A}) \longleftarrow$ under $\mathbb{D}$-colimits.
- $\mathbb{D}\text{-Ind}(\mathcal{A}) \leftarrow$ free cocompletion under $\mathbb{D}$-filtered colimits.
- $\mathcal{P}\mathcal{A} \longleftarrow$ " " " all colimits

(when $\mathcal{A}$ is small $\mathcal{P}\mathcal{A} = [\mathcal{A}^{op}, Set]$)

Prop: Let $\mathbb{D}$ be a doctrine; TFAE:

i) $\mathbb{D}$ is sound;

ii) $\forall$ small $\mathcal{A}$: $\mathcal{P}\mathcal{A} \simeq \mathbb{D}\text{-Ind}(\mathbb{D}(\mathcal{A}))$

iii) $\mathbb{D}\text{-Ind}(\_)$ restricts to

$$\begin{tikzcd} Cat \arrow[r, "\mathbb{D}\text{-Ind}"] & \mathbb{D}\text{-filt-}Cat \\ \mathbb{D}\text{-coc}\,Cat \arrow[u, hook] \arrow[r, dashed] & Coc\text{-}Cat \arrow[u, hook] \end{tikzcd}$$

Cats with $\mathbb{D}$-filt colimits and $\mathbb{D}$-filt. col. preserving functors

$\mathbb{D}$-cocomplete cats + $\mathbb{D}$-cocontinuous functors

cocomplete cats + cocontinuous functors

(iii) implies that if $\mathbb{D}$ is sound

$$\frac{\mathcal{A} \longrightarrow K \quad \mathbb{D}\text{-cocontinuous}}{\mathbb{D}\text{-Ind}(\mathcal{A}) \longrightarrow K \quad \text{cocontinuous}} \quad , K \text{ cocomplete}$$

Given $\mathbb{D}$ sound, we can describe $\mathbb{D}\text{-Ind}(\mathcal{A})$ explicitly:

$$\mathbb{D}\text{-Ind}(\mathcal{A}) \hookrightarrow [\mathcal{A}^{op}, Set]$$
$$\rotatebox{90}{$\in$} \; F$$

$$\underline{F \in \mathbb{D}\text{-Ind}(\mathcal{A})} \iff \underline{El(F)^{op} \text{ is } \mathbb{D}\text{-filtered}}$$

i.e F is $\mathbb{D}$-flat (def)

<u>Prop</u>: $\mathbb{D}$ is strongly sound iff : $\forall$ $F : \mathcal{C} \longrightarrow Set$

whenever $Lan_y F : [\mathcal{C}^{op}, Set] \longrightarrow Set$ preserves $\mathbb{D}$-limits of representables

$\implies$ F is $\mathbb{D}$-flat.

<u>Prop</u>: $\mathbb{D}$ is sound iff : $\forall$ $F : \mathcal{C} \longrightarrow Set$, $\mathcal{C}$ $\mathbb{D}$-complete

whenever $[Lan_y F : \underline{[\mathcal{C}^{op}, Set] \longrightarrow Set \text{ preserves } \mathbb{D}\text{-limits of representables}}]$

$\implies$ F is $\mathbb{D}$-flat. (F is $\mathbb{D}$-continuous)

From now on $\mathbb{D}$ is a sound doctrine:

Def: Let $\mathcal{A}$ be a cat, $A \in \mathcal{A}$, we say that A is $\mathbb{D}$-presentable if $\mathcal{A}(A, -)$ preserves $\mathbb{D}$-filtered colimits. $(A \in \mathcal{A}_{\mathbb{D}})$

Def: $\mathcal{A}$ is locally $\mathbb{D}$-presentable if it is cocomplete and has a small strong generator $\mathcal{G} \subseteq \mathcal{A}$ made of $\mathbb{D}$-presentable objects.

Prop: t.f.a.e:

i) $\mathcal{A}$ is l.$\mathbb{D}$.presentable ;

ii) $\mathcal{A} \simeq \mathbb{D}\text{-cont}(\mathcal{C}, \underline{Set})$, $\mathcal{C}$ $\mathbb{D}$-complete $\quad (\mathcal{C} = \mathcal{A}_{\mathbb{D}}^{op})$

iii) $\mathcal{A}$ is the cat of models of a sketch with $\mathbb{D}$-limits / no colimits.

iv) $\mathcal{A} \underset{\text{f.f and } \mathbb{D}\text{-filtered colimit preserving.}}{\overset{\perp}{\hookrightarrow}} [\mathcal{C}, Set]$

Theorem (G.U.)

$$\begin{array}{ccc} & & \text{locally } \mathbb{D}\text{-presentable} \\ \mathcal{C} & \longmapsto & \mathbb{D}\text{-Cont}(\mathcal{C}, Set) \\ \mathbb{D}\text{-cont-Cat} & \underset{\simeq}{\rightleftarrows} & LDP \\ \mathcal{A}_{\mathbb{D}}^{op} & \longleftarrow\!\shortmid & \mathcal{A} \end{array}$$

is a biequivalence.

– $\mathbb{D}$-Accessible cats.

Def: Say that $\mathcal{A}$ is $\mathbb{D}$-accessible if it has $\mathbb{D}$-filtered colimits and has $\mathcal{G} \subseteq \mathcal{A}_{\mathbb{D}}$ small which generates $\mathcal{A}$ under $\mathbb{D}$-filtered colimits.

Equivalently $\mathcal{A} \simeq \mathbb{D}\text{-Ind}(\mathcal{C})$ for some small $\mathcal{C}$.

Prop: TFAE:

(i) $\mathcal{A}$ $\mathbb{D}$-accessible

(ii) $\mathcal{A} \simeq \mathbb{D}\text{-Ind}(\mathcal{C})$, $\mathcal{C}$ small

(iii) $\mathcal{A} \simeq \mathbb{D}\text{-Flat}(\mathcal{C}, \mathcal{S}et)$ for some $\mathcal{C}$ small

(i) $\Rightarrow$ (iv) $\mathcal{A}$ is the category of models of a $\mathbb{D}$-limit/colimit sketch.

$\cancel{\Leftarrow}$ doesn't hold.

Theorem (ABLR): TFAE

i) $\mathcal{A}$ is accessible;

ii) $\mathcal{A}$ is $\mathbb{D}$-accessible $\boxed{\text{for some}}$ (strongly) sound doctrine $\mathbb{D}$.

The End

if $\mathcal{C}$-colimits commute in $\underline{\mathcal{S}et}$ with α-small products

(Adamek) $\Rightarrow$ they commute with all α-small limits

$\Rightarrow$ $\mathcal{C}$ is α-filtered

$\mathbb{D} \subseteq \mathbb{D}'$

$\mathbb{D}\text{-filt} \supseteq \mathbb{D}'\text{-filt}$

$\mathcal{A}$, $\mathcal{A}_{\mathbb{D}} \subseteq \mathcal{A}_{\mathbb{D}'}$ ($\mathcal{A}(A, -)$ preserves $\mathbb{D}'$-filt $\Leftarrow$ pres. $\mathbb{D}$-filt)

$(\mathbb{D}\subseteq\mathbb{D}') \Rightarrow \mathcal{A}$ is locally $\mathbb{D}$-presentable $\Rightarrow$ $\mathcal{A}$ is loc. $\mathbb{D}'$-presentable

Not true that: $\alpha < \beta \Rightarrow \mathcal{A}$ α-acc. implies $\mathcal{A}$ β-accessible

true: $\underline{\alpha \triangleleft \beta} \overset{\text{def}}{\Longleftrightarrow} \forall\ \lambda<\beta \quad \mathrm{cof}(\lambda^{\alpha}) < \beta$

(Ad-Ros)

$\mathbb{D} \rightsquigarrow \Phi = \{\varphi : \mathcal{C} \to \mathcal{V}\}$

Soundness: $\psi : \mathcal{C} \to \mathcal{V}$ Φ-continuous, $\mathcal{C}$ Φ-complete

$\Downarrow$

$\psi * \_ : [\mathcal{C}^{op}, \mathcal{V}] \to \mathcal{V}$ Φ-continuous

accessible $A = \bigcup_{\beta} A_{\beta}$

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd}$$

(α common for every)

profunctor $\nearrow$ Lex $\rightleftarrows$ LfP $\leftarrow$ filt. colim. pres.

[Garner-Power]

(Small Cat + ...) $\rightleftarrows$ $\mathbb{D}$-Acc $\swarrow$ $\mathbb{D}$-acc $\mathcal{A}$

$\mathbb{D}$-filt col.

some prop.

$\wr$

$$\left\{ \begin{array}{c} \text{Presheaves cats} \\ + \\ \mathbb{D}\text{-cont} + \text{coct functor} \end{array} \right\}$$

$\cong$