

# Accessible Categories made Sound

[ABLR]: Adamek-Borceux-Lack-Rosicky

- $A$  is accessible  $\stackrel{\text{def}}{\iff} A$  is  $\alpha$ -accessible for some regular cardinal  $\alpha$
- $A$  is  $\alpha$ -accessible if it is the free cocompletion of a small cat  $\mathcal{C}$  under  $\alpha$ -filtered colimits:  $A \cong \alpha\text{-Ind}(\mathcal{C})$

$$\left( \begin{array}{l} A \cong A_\alpha \text{ which generates under } \alpha\text{-filtered colimits} \\ \mathcal{C} = A_\alpha^{\text{op}} \end{array} \right)$$

Why is this important?, and the theory works  $\mathcal{C}^\infty$

$$D_\alpha = \{\alpha\text{-small categories}\}, \quad D_\omega = \{\text{finite categories}\} \quad \left\{ \begin{array}{l} \text{shapes of} \\ \text{the limits} \\ \text{you want to} \\ \text{consider} \end{array} \right.$$

$\rightarrow$   $\alpha$ -filtered colimits are those that commute with  $\alpha$ -small limits  
i.e. Set

$\rightarrow$  (Soundness)  $\mathcal{C}$  is  $\alpha$ -cocomplete  $\Rightarrow$  it is  $\alpha$ -filtered

• The General Setting:

Let  $\mathbb{D}$  be a doctrine: a small set of categories

- a  $\mathbb{D}$ -limit in a  $\mathcal{A}$  is the limit of  $H: \mathbb{D} \rightarrow \mathcal{A}$  where  $\mathbb{D} \in \mathbb{D}$ .

Def: Say that a small category  $\mathcal{C}$  is  $\mathbb{D}$ -filtered if  $\mathcal{C}$ -colimits commute in Set with  $\mathbb{D}$ -limits.

(def of [ABLR])

Def:  $\mathbb{D}$  is strongly sound: if  $\mathcal{C}$  is s.t.:  $\forall F: \mathbb{D}^{\text{op}} \rightarrow \mathcal{C}, \mathbb{D} \in \mathbb{D}$

the category of cocones of  $F$  is connected

obj:  $\eta: F \rightarrow \Delta A$

morph:

$$\begin{array}{ccc}
 F & \xrightarrow{\eta} & \Delta A & & A \\
 & \searrow & \downarrow \Delta f & & \downarrow f \\
 & & \Delta B & & B
 \end{array}$$

Then  $\mathcal{C}$  is  $\mathbb{D}$ -filtered

Def: We say that  $\mathbb{D}$  is sound if  $\forall \mathcal{C}$ : if  $\mathcal{C}$  is  $\mathbb{D}$ -cocomplete  
 $\Rightarrow$   $\mathcal{C}$  is  $\mathbb{D}$ -filtered.

Strongly sound  $\Rightarrow$  sound (Prop 2.5 in [ABLR])

Examples: The following are sound doctrines

$\mathbb{D}$ -filtered

i)  $\mathbb{D}_\alpha, \mathbb{D}_\alpha = \{\alpha\text{-small cats}\}$  ( $\alpha$  reg. card)  $\iff \alpha$ -filtered

ii)  $\mathbb{D}_{\text{pr}} = \{\text{finite discrete categories}\}$   $\iff$  sifted cat

iii)  $\mathbb{D}_{\text{con}} = \{\text{finite connected categories}\}$  ?

iv)  $\mathbb{D}_{\text{terru}} = \{\emptyset\}$   $\iff$  (poss. infinite) connected cats

v)  $\emptyset$   $\iff$  all cats

vi)  $\mathbb{D}_{\text{weird}} = \{\emptyset\} \cup \{\rightarrow \downarrow\}$  is sound but not strongly sound.

[Conjecture: If  $\mathbb{D}$  is "saturated"  $\Rightarrow$  sound = strongly sound]

Def: Let  $\mathbb{D}$  be a class of indexing categories; given  $\mathcal{A}$  we denote by  $\mathbb{D}(\mathcal{A})$  the free cocompletion of  $\mathcal{A}$  under  $\mathbb{D}$ -colimits

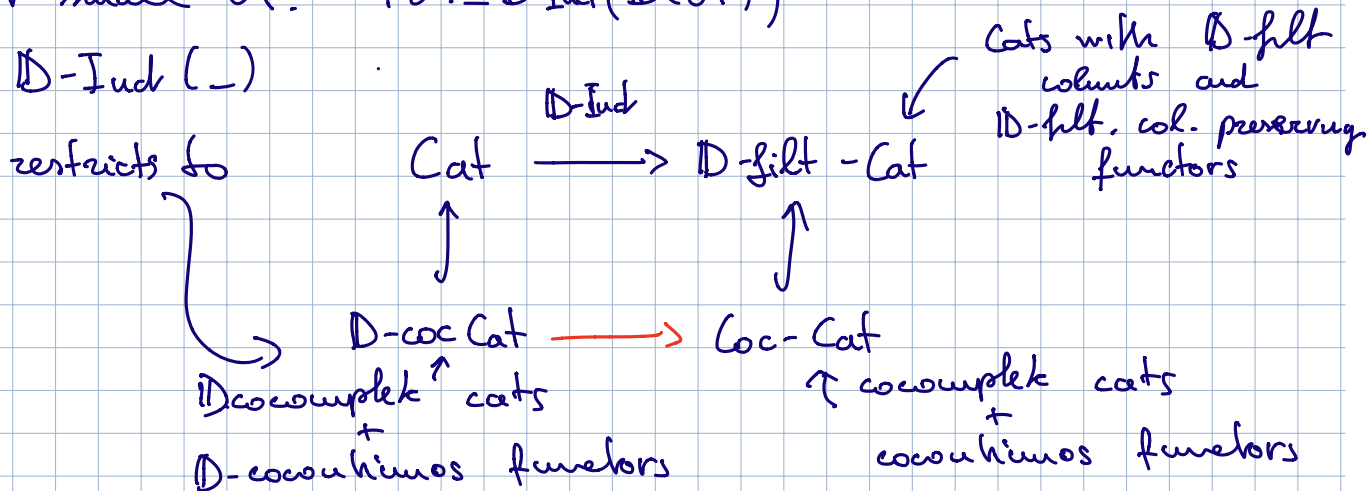
$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{K} \\
 \text{universal prop.} & \xrightarrow{\quad} & \uparrow \mathbb{D}\text{-cocomplete} \\
 \mathcal{A} \xrightarrow{J} \mathbb{D}\mathcal{A} \in [A^{op}, \text{Set}] & & \mathbb{D}\mathcal{A} \xrightarrow{\text{Lan}_J F} \mathcal{K} \text{ } \mathbb{D}\text{-cocomplete} \\
 \text{↳ the closure of the reps. in } [A^{op}, \text{Set}] \text{ under } \mathbb{D}\text{-colimits.} & & 
 \end{array}$$

Given a doctrine  $\mathbb{D}$ , can consider

- $\mathbb{D}(\mathcal{A}) \leftarrow$  under  $\mathbb{D}$ -colimits.
- $\mathbb{D}\text{-Ind}(\mathcal{A}) \leftarrow$  free cocompletion under  $\mathbb{D}$ -filtered colimits.
- $\mathcal{P}\mathcal{A} \leftarrow$  " " " all colimits  
(when  $\mathcal{A}$  is small  $\mathcal{P}\mathcal{A} = [A^{op}, \text{Set}]$ )

Prop: Let  $\mathbb{D}$  be a doctrine;  $f, g$  are:

- $\mathbb{D}$  is sound;
- $\forall$  small  $\mathcal{A}$ :  $\mathcal{P}\mathcal{A} \simeq \mathbb{D}\text{-Ind}(\mathbb{D}(\mathcal{A}))$
- $\mathbb{D}\text{-Ind}(-)$



(iii) implies that if  $\mathbb{D}$  is sound

$$\frac{A \longrightarrow K \quad \mathbb{D}\text{-cocomplete, } K\text{ cocomplete}}{\mathbb{D}\text{-Ind}(A) \longrightarrow K \quad \text{cocomplete}}$$

Given  $\mathbb{D}$  sound, we can describe  $\mathbb{D}\text{-Ind}(A)$  explicitly:

$$\mathbb{D}\text{-Ind}(A) \xleftrightarrow{\quad} [A^{\text{op}}, \text{Set}]$$

$\Psi$   
 $F$

$$\underline{F \in \mathbb{D}\text{-Ind}(A)} \iff \underline{\text{El}(F)^{\text{op}} \text{ is } \mathbb{D}\text{-filtered}}$$

i.e.  $F$  is  $\mathbb{D}$ -flat (def)

Prop:  $\mathbb{D}$  is strongly sound iff:  $\forall F: \mathcal{C} \rightarrow \text{Set}$

whenever  $\text{Lan}_y F: [\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow \text{Set}$  preserves  $\mathbb{D}$ -limits of representables  
 $\Rightarrow F$  is  $\mathbb{D}$ -flat.

Prop:  $\mathbb{D}$  is sound iff:  $\forall F: \mathcal{C} \rightarrow \text{Set}$ ,  $\mathcal{C}$   $\mathbb{D}$ -cocomplete

whenever  $\left[ \text{Lan}_y F: [\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow \text{Set} \text{ preserves } \mathbb{D}\text{-limits of representables} \right]$   
 $\Rightarrow F$  is  $\mathbb{D}$ -flat. ( $F$  is  $\mathbb{D}$ -continuous)

—————  $\circ$  —————

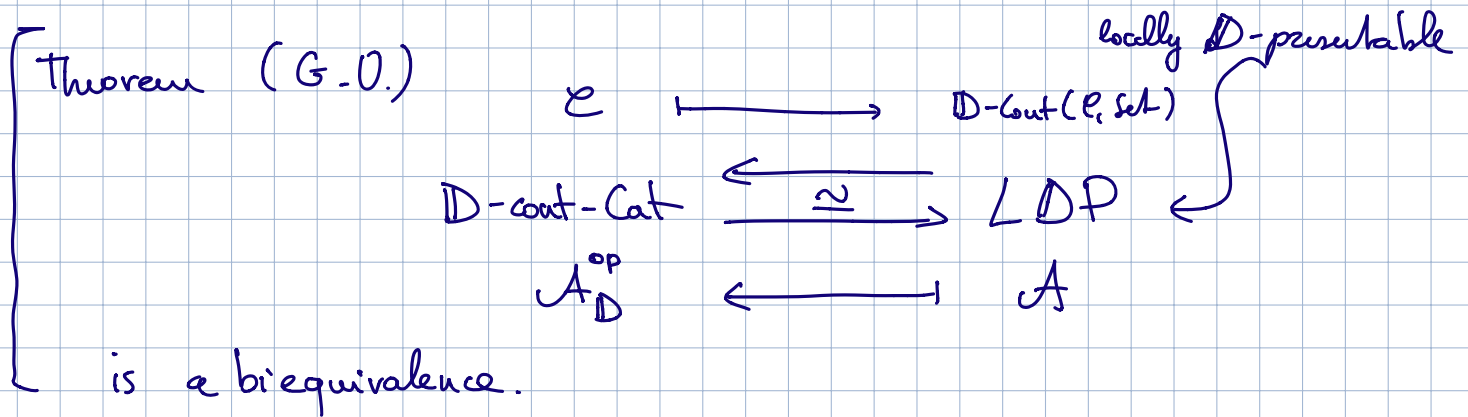
From now on  $\mathbb{D}$  is a sound doctrine:

Def: Let  $\mathcal{A}$  be a cat,  $A \in \mathcal{A}$ , we say that  $A$  is  $\mathbb{D}$ -presentable if  $\mathcal{A}(\mathcal{A}, -)$  preserves  $\mathbb{D}$ -filtered colimits. ( $A \in \mathcal{A}_{\mathbb{D}}$ )

Def:  $\mathcal{A}$  is locally  $\mathbb{D}$ -presentable if it is cocomplete and has a small strong generator  $\mathcal{G} \subseteq \mathcal{A}$  made of  $\mathbb{D}$ -presentable objects.

Prop: t.f.a.e:

- i)  $\mathcal{A}$  is l. $\mathbb{D}$ .presentable ;
- ii)  $\mathcal{A} \simeq \mathbb{D}\text{-catt}(\mathcal{C}, \underline{\text{Set}})$ ,  $\mathcal{C}$   $\mathbb{D}$ -complete ( $\mathcal{C} = \mathcal{A}_{\mathbb{D}}^{\text{op}}$ )
- iii)  $\mathcal{A}$  is the cat of models of a sketch with  $\mathbb{D}$ -limits / no colimits.
- iv)  $\mathcal{A} \xleftarrow[\text{f.f. and } \mathbb{D}\text{-filtered colimit preserving.}]{\perp} [\mathcal{C}, \underline{\text{Set}}]$



- $\mathbb{D}$ -Accessible cats.

Def: Say that  $\mathcal{A}$  is  $\mathbb{D}$ -accessible if it has  $\mathbb{D}$ -filtered colimits and has  $\mathcal{G} \subseteq \mathcal{A}_{\mathbb{D}}$  small which generates  $\mathcal{A}$  under  $\mathbb{D}$ -filtered colimits.

Equivalently  $\mathcal{A} \simeq \mathbb{D}\text{-Ind}(\mathcal{C})$  for some small  $\mathcal{C}$ .

Prop: TFAE:

(i)  $A$   $D$ -accessible

(ii)  $A \simeq D\text{-Ind}(\mathcal{C})$ ,  $\mathcal{C}$  small

(iii)  $A \simeq D\text{-Flat}(\mathcal{C}, \text{Set})$  for some  $\mathcal{C}$  small

(i)  $\Rightarrow$  (iv)  $A$  is the category of models of a  $D$ -limit/colimit sketch.

~~$\Leftarrow$~~  doesn't hold.

Theorem (ABL): TFAE

i)  $A$  is accessible;

ii)  $A$  is  $D$ -accessible for some  $(\text{strongly})$  doctrine  $D$ .

The End

if  $\mathcal{C}$ -colimits commute in Set with  $\alpha$ -small products

(Adamek)

$\Rightarrow$  they commute with all  $\alpha$ -small limits

$\Rightarrow \mathcal{C}$  is  $\alpha$ -filtered

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$$D \subseteq D'$$

$$D\text{-filt} \supseteq D'\text{-filt}$$

$$A, \quad A_D \subseteq A_{D'} \quad (A(A, -) \text{ preserves } D'\text{-filt} \Leftarrow \text{pres. } D\text{-filt})$$

$(\mathbb{D} \leq \mathbb{D}') \Rightarrow A$  is locally  $\mathbb{D}$ -presentable  $\Rightarrow A$  is loc.  $\mathbb{D}'$ -presentable

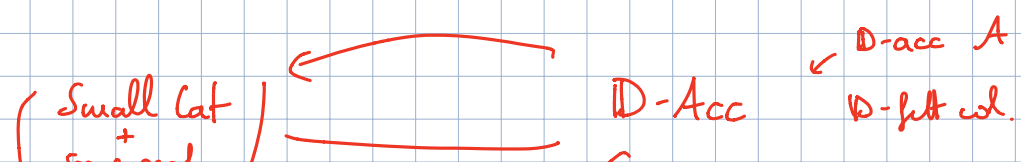
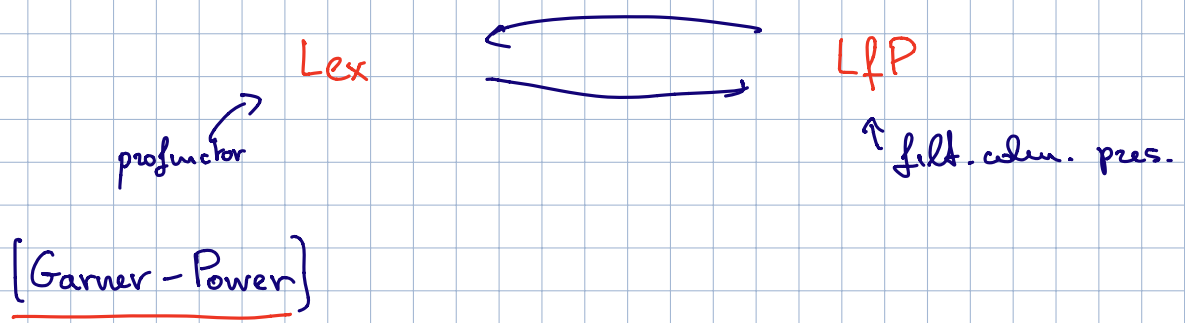
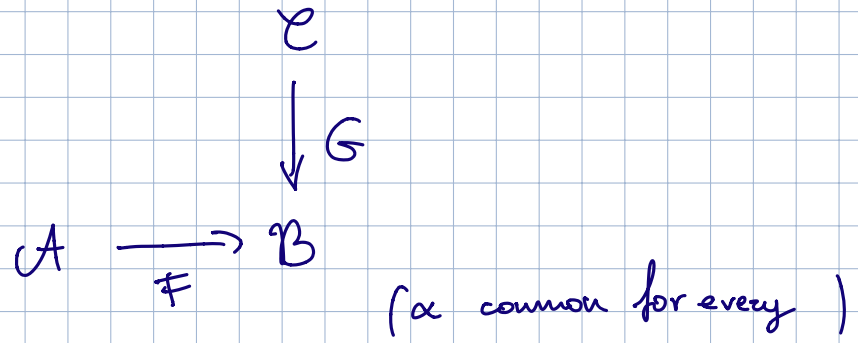
Not true that:  $\alpha < \beta \Rightarrow A$   $\alpha$ -acc implies  $A$   $\beta$ -accessible

true:  $\frac{\alpha < \beta}{(Ad-Rox)}$   $\overset{dir}{\rightleftarrows} \forall \lambda < \beta \text{ cof}(\lambda^\alpha) < \beta$

$\mathbb{D} \rightsquigarrow \Phi = \{ \varphi: e \rightarrow V \}$

Soundness:  $\psi: e \rightarrow V$   $\Phi$ -continuous,  $e$   $\Phi$ -complete  
 $\Downarrow$   
 $\psi * \_ : [e^\varphi, V] \rightarrow V$   
 $\Phi$ -continuous

accessible  $A = \bigcup_{\beta} A_{\beta}$



some prog.  
12  
Presheaves cats  
+  
D-coat + coct functor

