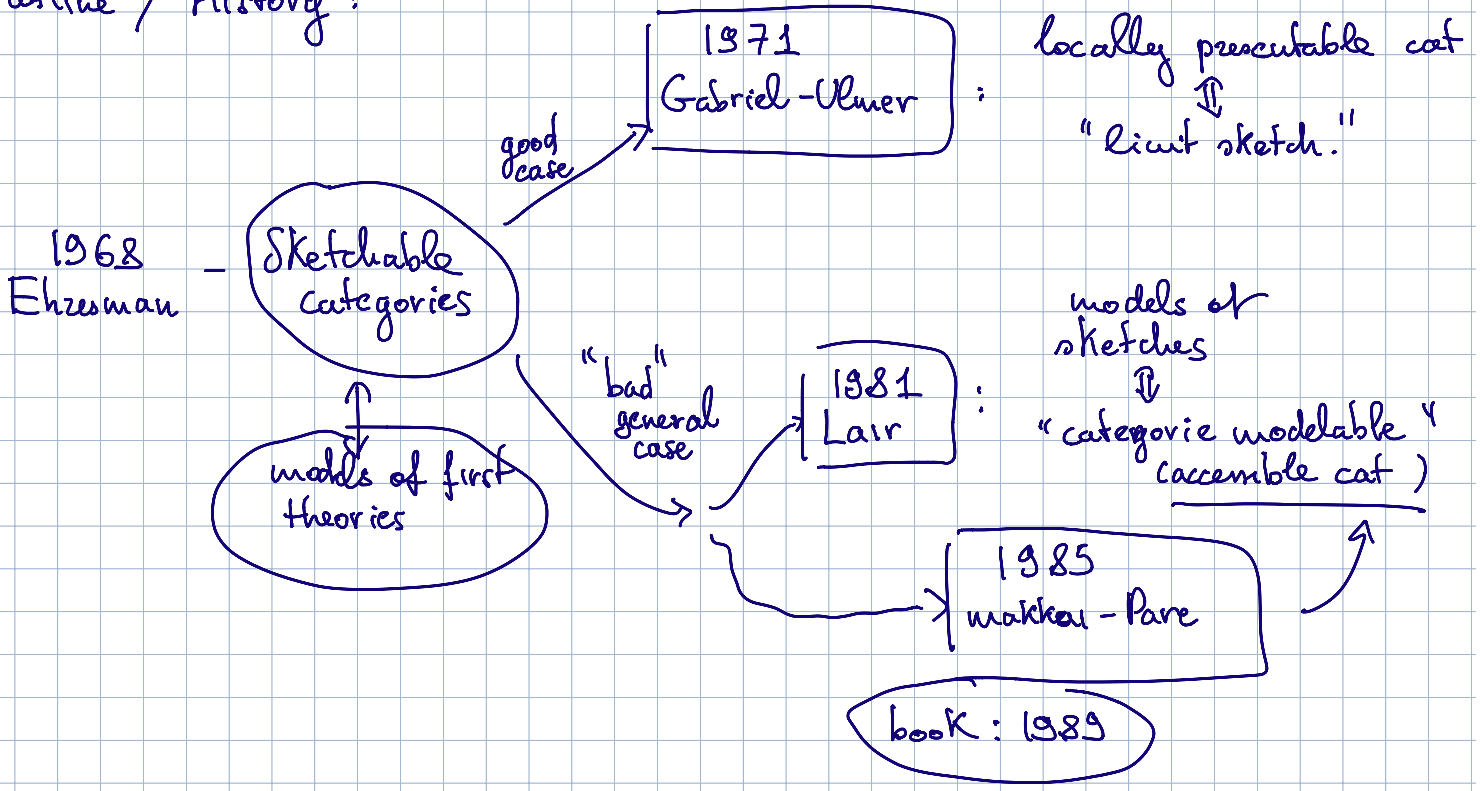
# Accessible Categories made Accessible

(thanks Adrian)

## 0) Outline / History :



## I) Sketches :

**Def:** A sketch is the data of $S = (\mathcal{C}, \mathbb{L}, \mathbb{C})$ where:

i) $\mathcal{C}$ is a small category

ii) $\mathbb{L}$ is a set of cones in $\mathcal{C}$ : $\mathbb{L} \ni \left\{ \begin{array}{ll} H : \overset{\text{small}}{\mathcal{E}} \longrightarrow \mathcal{C} & \text{diagram} \\ \eta : \Delta A \longrightarrow H & \text{cone} \end{array} \right.$

iii) $\mathbb{C}$ is a set of cocones in $\mathcal{C}$ : $\mathbb{C} \ni \left\{ \begin{array}{ll} K : \overset{\text{small}}{\mathcal{F}} \longrightarrow \mathcal{C} & \text{diagram} \\ \gamma : K \longrightarrow \Delta B & \text{cocone} \end{array} \right.$

- Model of $S$: $F : \mathcal{C} \longrightarrow Set$ functor s.t.

  i) $\forall\ (\eta, H) \in \mathbb{L}$ $\quad F\eta$ is a limiting cone for $FH$

  $( FA \cong \lim FH )$

ii) $\forall\ (\gamma, K) \in \mathbb{C}$ $\quad F\gamma$ is a colimiting cone for $FK$
$(FB \cong \operatorname{colim} FK)$

- $\mathrm{Mod}(S) \subseteq [\mathcal{C}, \mathrm{Set}]$ is the full subc. of models of $S$.

## II) (finite) Product Sketches : $\mathbb{C} = \emptyset$, $\mathbb{L}$ has only discrete diagrams

Example: Sketch for semigroups

$$\mathcal{C} : \left\{ \begin{tikzcd} S^3 \arrow[r, "q_1{,}q_2"] \arrow[r, bend left=50, "m\times\pi_3{,}\ \pi_1\times m"] \arrow[dr, bend right, "p_1{,}p_2{,}p_3"'] & S^2 \arrow[d, "m"] \arrow[d, bend right=50, "\pi_1"'] \arrow[d, bend left=50, "\pi_2"] \\ & S \end{tikzcd} \ \middle| \ \begin{tikzcd} S^3 \arrow[r, "m\times\pi_1"] \arrow[d, "\pi_1\times m"'] \arrow[dr, phantom, "\circlearrowleft"] & S^2 \arrow[d, "m"] \\ S^2 \arrow[r, "m"'] & S \end{tikzcd}\ ,\ \begin{tikzcd} S^3 \arrow[r, "m\times\pi_3"'] \arrow[d, "q_1"'] \arrow[dr, phantom, "\circlearrowleft"] & S^2 \arrow[d, "\pi_1"] \\ S^2 \arrow[r, "m"'] & S \end{tikzcd}\ , \ldots \right\}$$

$m\times\pi_3$ is what it is

$\mathbb{C} = \emptyset$

$$\mathbb{L} = \left\{ \begin{tikzcd} & S^2 \arrow[dl] \arrow[dr] & \\ S & & S \end{tikzcd}\ ,\ \begin{tikzcd} & S^3 \arrow[dl] \arrow[d] \arrow[dr] & \\ S & S & S \end{tikzcd} \right\}$$

$\mathrm{Mod}(S) \ni F : \mathcal{C} \to \underline{\mathrm{Set}}$ $\quad F(S)$ is a set $\leftarrow$ has a semigroup structure.
$F(S^2) = F(S) \times F(S)$ ....

- $\mathrm{Mod}(S) \simeq \mathrm{sGrp}$
- $\rightsquigarrow$ Modify $\mathcal{C}$ : add a "0" $\rightsquigarrow$ $\mathrm{Mod}(S') = \mathrm{Monoids}$
- $\rightsquigarrow$ $M(S'') = \mathrm{Rings}$

Models of product Sketches $\Leftrightarrow$ Models of Algebraic theories
$\Leftrightarrow$ finitary varieties.

## III) Limit Sketches: $\mathbb{C} = \emptyset$

Example: Sketch for Cat

$\mathcal{C}$ :
$$\begin{tikzcd}
a_1\times_0 a_1 \arrow[d, bend right, "\pi_1"'] \arrow[d, "m"] \arrow[d, bend left, "\pi_2"] & a_1\times_0 a_1\times_0 a_1 \arrow[l, bend right, "m\times 1"'] \arrow[l, "1\times m"] \arrow[dl] \arrow[dl, bend left=20] \arrow[dl, bend left=40] \\
a_1 \arrow[d, bend right, "t,s"'] & \\
a_0 \arrow[u, "1"'] &
\end{tikzcd}$$

commutativity conditions:

- $$\begin{tikzcd}
a_0 \arrow[r] \arrow[rr, bend left=50, "1"] & a_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & a_0
\end{tikzcd}$$
- associativity

$\mathbb{L}$ :
$$\begin{tikzcd}
a_1\times_0 a_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & a_1 \arrow[d, "t"] \\
a_1 \arrow[r, "s"'] & a_0
\end{tikzcd}
,
\begin{tikzcd}
 & & a_1\times_0 a_1\times_0 a_1 \arrow[dll] \arrow[d] \arrow[drr] & & \\
a_1 \arrow[dr, "s"'] & & a_1 \arrow[dl, "t"] \arrow[dr, "s"'] & & a_1 \arrow[dl, "t"] \\
 & a_0 & & a_0 &
\end{tikzcd}$$

$\mathrm{Mod}(S) \simeq$ Cat

- Let $S = (\mathcal{C}, \mathbb{L})$ be a limit sketch

$$\begin{tikzcd}
\mathrm{Mod}(S) \arrow[r, hook, "J"'] \arrow[r, phantom, shift left=2, "\perp"] & {[\mathcal{C}, \mathrm{Set}]} \arrow[l, bend right=40, "L"']
\end{tikzcd}$$

is complete and cocomplete

J is continuous

Note: J preserves all colimits that commute with the limits in S.

Consider the finite limits case: $\mathbb{L}$ has diagrams with finite domain
(e.g. the sketch for Cat)

$$\begin{tikzcd} \mathrm{Mod}(S) \arrow[r, hook, "J"'] \arrow[r, phantom, shift left=2, "\perp"] & {[\mathcal{C},\underline{\mathrm{Set}}]} \arrow[l, bend right, "L"'] \end{tikzcd}$$

$\uparrow$ J is continuous

- J preserves <u>all colimits that commute with finite limits in Set</u>

$$\| $$

<u>filtered colimits</u>

Def: A $\overset{\text{small}}{\wedge}$ category $\mathcal{E}$ is filtered if $\mathcal{E}$-colimits commute with finite limits in $\underline{\mathrm{Set}}$.

Let's characterize filtered categories $\mathcal{E}$:

- $\mathcal{E}$-colimits commute with finite products:

$$\forall\ e \in \mathcal{E} \quad \mathrm{colim}\left(\mathcal{E}(e,-) : \mathcal{E} \longrightarrow \underline{\mathrm{Set}}\right) \cong 1 \quad (\text{for any } \mathcal{E})$$

now: $\forall\ d,e \in \mathcal{E}$

$$\mathrm{colim}\left(\mathcal{E}(d,-) \times \mathcal{E}(e,-)\right) \cong$$

$$\cong \left(\mathrm{colim}\ \mathcal{E}(d,-)\right) \times \left(\mathrm{colim}\ \mathcal{E}(e,-)\right)$$

$$= 1 \times 1 = 1$$

is not $\emptyset$

$$\Rightarrow \exists\ f \in \mathcal{E} \text{ s.t. } \mathcal{E}(d,f) \times \mathcal{E}(e,f) \neq \emptyset$$

$\Rightarrow$ given $d,e$ $\exists\ f$ and maps $\begin{tikzcd}[row sep=small] d \arrow[dr] & \\ & f \\ e \arrow[ur] & \end{tikzcd}$

①

- Given any $d \overset{h}{\underset{k}{\rightrightarrows}} e$ in $\mathcal{E}$

$$\mathrm{colim}\left(\mathrm{Eq}\left(\mathcal{E}(h,-), \mathcal{E}(k,-)\right)\right)$$

$$\cong$$

$$\mathrm{Eq}\Big(\underset{\mathbb{1}}{\underset{\shortparallel}{\mathrm{colim}\,\mathcal{E}(h,-)}}\ ,\ \underset{\mathbb{1}}{\underset{\shortparallel}{\mathrm{colim}\,\mathcal{E}(k,-)}}\Big)$$

$$= \mathbb{1}$$

not $\emptyset$ $\Rightarrow \exists\, x \in \mathcal{E}$ s.t. $\mathrm{Eq}\big(\mathcal{E}(h,x), \mathcal{E}(k,x)\big) \neq \emptyset$

② $\forall\, h,k : d \to e \quad \exists \quad e \to x$

$$\begin{tikzcd} \text{s.t.} & d \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & e \arrow[r] & x & \text{coincide} \end{tikzcd}$$

- If $\mathcal{E} \overset{\text{bad filtered}}{=} \emptyset \Rightarrow \mathbb{1} = \mathrm{colim}\,(\underset{\mathbb{1}}{\underset{\shortparallel}{\lim \emptyset}}) \cong \lim\,(\underset{0}{\underset{\shortparallel}{\mathrm{colim}\,\emptyset}}) = 0$. contradiction

③ $\mathcal{E}$ filtered $\Rightarrow \mathcal{E} \neq \emptyset$

- this three conditions are enough to imply that $\mathcal{E}$ is filtered.

Consider the finite limits case: $\mathbb{L}$ has diagrams with finite domain

(e.g. the sketch for Cat)

$\mathcal{G} \subseteq K$ is a strong generator if:

$\forall\, f : X \to Y$ in $K$

$f$ is iso $\Leftrightarrow K(G,f)$ is iso $\forall\, G \in \mathcal{G}$

$\Leftrightarrow K \xrightarrow[K(\mathcal{G},1)]{\text{conservative}} [\mathcal{G}^{op}, Set]$

$$\begin{tikzcd} L\mathcal{G} \arrow[rr, bend left, <->] & & {} \\ \mathrm{Mod}(S) \arrow[r, shift right, "J"'] \arrow[r, phantom, "\perp"] & {[\mathcal{C}, Set]} \arrow[l, shift right, bend right, "L"'] & \supseteq Y\mathcal{C}^{op} = \mathcal{G} \end{tikzcd}$$

$J$ is continuous and preserves filtered colimits.

- $L\mathcal{G}$ is a strong generator of $\mathrm{Mod}(S)$

and is made of finitely presentable objects:

$X \in L\mathcal{G} \qquad \mathrm{Mod}(S)(X,\_)$ preserves filt. colimits

$$\underset{\cong}{\underset{\mathrm{Mod}(S)(LG,\_)}{\underset{\shortparallel}{\phantom{x}}}}$$

$[\mathcal{C}, Set](G,\_)$ is cocontinuous

$[\mathcal{C}, Set](G, J\_)$ ← and $J$ preserves filt. colimt

- Mod($\mathcal{S}$) : Is cocomplete and has a strong generator of finitely presentable objects

!! G.U

locally finitely presentable category

Models of finite limits sketches ⟺ locally finitely presentable cat.
($\alpha$) ($\alpha$)

Models of limit sketches $\overset{G.U}{\Longleftrightarrow}$ locally presentable cats.

## III) (Mixed) Sketches

example 1

$$S \begin{cases} \mathcal{C} = \mathbb{2} = \{0 \xrightarrow{!} 1\} \\ \mathbb{C} = \left\{ ! \; \begin{tikzcd} 0 \arrow[r, "!"] \arrow[d] & 1 \arrow[d, "1"] \\ 1 \arrow[r, "1"'] & 1 \end{tikzcd} \right\} \\ \mathbb{L} = \{ \text{empty diagram over } 1 \} \end{cases}$$

Mod $(S)$ : $F : \mathbb{2} \longrightarrow Set$

- $F(1) = 1$
- $F(0) = A$
- $A \twoheadrightarrow 1$ epimorphism.

$$\boxed{\text{Mod}(S) = \underline{Set}_{\neq \emptyset}}$$

example 2 : Fields

r: ring

i: multiplicative group

$$\mathcal{C} : \left\{ \begin{tikzcd} r^3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & r^2 \arrow[d, shift left] \arrow[d, shift right] & i^3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & i^2 \arrow[d, shift left] \arrow[d, shift right] \\ & r & & i \arrow[ll, "i"'] \\ & {} \arrow[u, "0"] \arrow[rrru, bend right, "1"'] & & \end{tikzcd} \right.$$

$\Bigg[$ diagrams need to commute $\Bigg\}$

....

( $Z$ )

$\mathbb{L}$ : $\left\{ \begin{array}{l} \text{product specifications ( for ring and the group}^{+} \text{)} \\ \text{empty diagram over } Z \end{array} \right\}$

$\mathbb{C}$ : $\left\{ \begin{tikzcd} Z \arrow[dr, "0"'] & & i \arrow[dl, "i'"] \\ & r & \end{tikzcd} \right\}$

$\boxed{\mathrm{Mod}(S) \simeq \text{Fields.}}$

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$S = (\mathcal{C}, \mathbb{L}, \mathbb{C})$ is a sketch, with finite limits specifications.

$$\begin{tikzcd} \mathcal{A} = \mathrm{Mod}(\mathcal{C}) \arrow[r, hook, "J"'] & {[\mathcal{C}, \mathcal{S}et]} \arrow[l, dashed, bend right, "\times"'] \end{tikzcd}$$

- $J$ preserves filtered colimits.
- $J$ satisfies the solution set condition

$$\left( \forall X \in [\mathcal{C}, \mathcal{S}et] \quad \sum_{i \in \underset{\substack{\uparrow \\ \text{small}}}{I}} \mathcal{A}(\underline{A_i}, \_) \xrightarrow[\text{epi}]{} K(X, J\_) \right)$$

$\mathcal{A}$ will have: a set $\{A_i\}_i$ s.t. $\exists \alpha$

i) $A_i$ is $\alpha$- presentable ( $\mathcal{A}(A_i, \_)$ preserves $\alpha$-filt. colimits )

ii) $\{A_i\}$ generate $\mathcal{A}$ under $\alpha$-filtered colimits

$\Rightarrow$ $\mathcal{A}$ is $\alpha$- accessible

Models of finite limit/colimit sketches $\overset{\Longleftarrow}{\underset{\not\Longrightarrow}{\neq}}$ finitely accessible categories

Models of sketches $\iff$ Accessible categories
($\alpha$-acc. for some $\alpha$)

Dependence on $\alpha$ is bad $\rightsquigarrow$ ABLR : soundness
(get rid of cardinals)

The End

$K$ is loc. fin. pres

$K_f \subseteq K$ $\qquad$ $K \simeq \mathrm{Lex}(K_f^{op}, \underline{Set})$

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$S$, $\quad \mathbb{L} = \emptyset$, $\mathbb{C}$ = general

$\Rightarrow$ $\mathrm{Mod}(S)$ is locally presentable (acc + cocomplete)

$$\Rightarrow \left( \begin{tikzcd} \mathrm{Mod}(S) = \overset{l.p}{K} \arrow[r, hook, "\overset{f.f.}{J}"] \arrow[r, phantom, shift right=3, "\perp"] & {[\mathbb{C}, Set]} \arrow[l, bend left=50] \end{tikzcd} \right)$$

cocontinuous

$$\begin{tikzcd} K \arrow[r, hook, "J"'] \arrow[r, phantom, shift left=3, "\perp"] & {[\mathbb{C}, Set]} \arrow[l, bend right=40, "L"'] \end{tikzcd}$$

$\left( LX := \{K(X, J\_), J\_\} \right)$