Stack Representation of Logical Theories

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For propositional logic we have the following picture:

$$\begin{array}{ccccccc}
\mathbf{BA}_{\text{f.}}^{\mathrm{op}} & \rightarrowtail & \mathbf{HA}_{\text{f.}}^{\mathrm{op}} & \rightarrowtail & \mathbf{DL}_{\text{f.}}^{\mathrm{op}} & \rightarrowtail & \mathbf{HA}_{\text{f.p.}}^{\mathrm{op}} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathbf{Fin} & \rightarrowtail & \mathbf{FKp} & \rightarrowtail & \mathbf{FPos} & \rightarrowtail & \mathbf{H}
\end{array}$$

Traditionally, dual objects are constructed as *petit spectrum*,

$$\mathsf{Spec}\, A = \{ \text{prime ideals on } A \cdots \} + \mathcal{O}_A.$$

In these cases, $\mathcal{O}_A$ can be recovered by additional topological data.

Exactness Property of the Dual Category

The dual category in some sense can be viewed as the category of *formulas* in the corresponding fragment of propositional logic $\mathbb{P}$:

$$P[x]/\varphi(x) \twoheadrightarrow P[x]/\psi(x) \Leftrightarrow \psi(x) \vdash_{\mathbb{P}} \varphi(x).$$

Thus we would like to understand exactness properties of the dual,

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Sheaf Representation for Heyting Algebras

One studies $\mathbf{H} \simeq \mathbf{HA}_{\text{f.p.}}^{\text{op}}$ via embedding it into a *gros topos*:

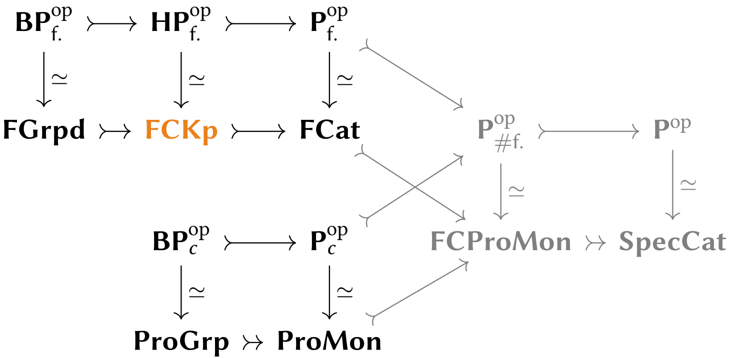
- The site is $(\mathbf{HA}_{\text{f.p.}}^{\text{op}}, J^{\infty})$ with J^{∞} = jointly monic families.
- By FMP, $\mathcal{H} := \mathbf{Sh}(\mathbf{HA}_{\text{f.p.}}^{\text{op}}, J^{\infty}) \simeq \mathbf{Sh}(\mathbf{HA}_{\text{f}}^{\text{op}}, J^{\text{can}})$.
- $\mathsf{Spec} : \mathbf{HA}^{\text{op}} \rightarrowtail \mathcal{H}$ essentially by Yoneda,

$$\mathsf{Spec}\, H = \mathbf{HA}(H, -) : \mathbf{HA}_{\text{f.p.}} \to \mathbf{Set}.$$

- From a logical perspective, $\mathsf{Spec}\, H$ is the sheaf of models:

$$\mathsf{Spec}\, H \cong \mathbf{Mod}_H(-) : \mathbf{HA}_{\text{f}} \simeq \mathbf{FKp}^{\text{op}} \to \mathbf{Set}.$$

For the first-order fragment:



Consider the naive embedding $\mathsf{Spec} : \mathbf{HPt}_{\text{f.p.}}^{\text{op}} \to \mathbf{St}(\mathbf{FCKp}, J^{\text{can}})$,

$$\mathsf{Spec}\, H = \mathbf{Mod}_H(-) = \mathbf{HPt}(H, [-, \mathbf{Set}]) : \mathbf{FCKp}^{\text{op}} \to \mathbf{CAT}.$$

- $\mathsf{Spec}\, H$ is a stack by open surjections being effective descent.
- Spec is faithful iff IFL has *finite frame property*.

Questions

- Concerning FFP, **FCKp** is indeed more faithful than **FKp**,

$$\mathbf{FKp} \models \forall x.\neg\neg P(x) \to \neg\neg\forall x.P(x).$$
$$\mathbf{FCKp} \not\models \forall x.\neg\neg P(x) \to \neg\neg\forall x.P(x)$$

- Would gros representation simplifies the dual theory for $\mathbf{P}_{\text{f.p.}}$?

Towards Stack Representation for Intuitionistic Theories

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