Directed univalence in simplicial homotopy type theory

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Simplicial homotopy type theory (sHoTT)

- **Goal:** develop synthetic ∞-category theory
- Compatibility with other approaches such as Riehl–Verity, Cisinski–Cnossen–Nguyen–Walde, Martini–Wolf
- **Setting:** Homotopy type theory extended by an **interval** $\mathbb{I}$, after Riehl–Shulman '17:
  $\rightsquigarrow$ shapes $\mathbb{I}^n$, Δ^n, $\partial\Delta^n$, Λ^n_k, ...

Definition (Categories and groupoids)

- A is a **pre-category** if $A^{\Delta^2} \xrightarrow{\simeq} A^{\Lambda^2_1}$ (after Joyal).
- A pre-category A is a **category** if $A \xrightarrow{\simeq} A^{\mathbb{E}}$
- A type A is a **groupoid** or **space** if $A \xrightarrow{\simeq} A^{\Delta^1}$.

Definition (hom-types)

If A is a type with terms $x, y : A$, then the type of **homomorphisms** is defined by:

$$\hom_A(x,y) :\equiv \Sigma_{f:\mathbb{I}\to A}(f\,0 = x) \times (f\,1 = y)$$

Synthetic ∞-category theory in sHoTT

- Functors, natural transformations, discrete fibrations & fibered Yoneda lemma, adjunctions (Riehl–Shulman '17)
- Cartesian fibrations (Buchholtz–W '21) & generalizations (W '21)
- Limits and colimits (Bardomiano '22)
- Conduché fibrations (Bardomiano '24)
- Proof assistant `Rzk` (Kudasov '23) and formalization of fibered Yoneda lemma (Kudasov–Riehl–W '23)
- sHoTT library and more formalizations (Abounegm, Bakke, Bardomiano, Campbell, Carlier, Chatzidiamantis-Christoforidis, Ergus, Hutzler, Kudasov, Maillard, Martínez, Pradal, Rasekh, Riehl, Verity, Walde, W '23–)

But many desiderata missing!

opposite categories, categories $\mathcal{S}$ *and* Cat*, presheaves & Yoneda embedding, higher algebra,*

$\ldots$

Modal operators

- **Multi-modal dependent type theory (MTT)** to the rescue! (Gratzer–Kavvos–Nuyts–Birkedal '20)
- add modal operators: **opposite** $\mathfrak{o}$, **core** $\flat$, **twisted arrows** $\mathfrak{t}$
- plus axioms about the interaction between the simplicial and modal structure
- This unlocks a whole new range of constructions...

$$\begin{array}{c} \mathcal{S}^{\Delta^{\mathrm{op}}} \\ \downarrow\uparrow\downarrow\uparrow \\ \mathcal{S} \end{array}$$

See also work on cohesive ∞-topoi by Schreiber ('13), Shulman ('18), Myers–Riley ('23), as well as internal universes via a tiny interval by Licata–Orton–Pitts-Spitters ('18) and Riley ('24).

Directed univalence

Theorem (Directed univalence for groupoids, Gratzer–W–Buchholtz '23)

1. *There is a category $\mathcal{S}$ whose terms are groupoids.*
2. *$\mathcal{S}$ classifies left fibrations.**
3. *$\mathcal{S}$ is closed under Σ, identity types, and (co)limits.*
4. *$\mathcal{S}$ is **directed univalent**:*

$$\text{arrtofun} : (\Delta^1 \to \mathcal{S}) \simeq \left(\sum_{A,B:\mathcal{S}} (A \to B) \right)$$

Also investigated in a bicubical setting by Weaver–Licata '20, but using some different methods.

Corollary (Directed structure identity principle (DSIP))

Over $\mathcal{S}$, types of algebraic structures have the correct notions of morphisms.

Corollary (Categories of algebraic structures)

We can define categories of algebraic structures such as: posets, reflexive graphs (hence Δ), ∞-monoids, and spectra.

Presheaf theory

The category $\mathcal{S}$ of spaces unlocks various constructions (forthcoming).

Let C be a category. Using the twisted arrow modality $\mathfrak{t}$, we obtain the hom-bifunctor $\Phi : C \times \langle \mathfrak{o} | C \rangle \to \mathcal{S}$ and presheaf categories $\mathrm{PSh}(C) :\equiv \langle \mathfrak{o} | C \rangle \to \mathcal{S}$.

Theorem (Yoneda lemma)

We have $\hom(\Phi(-, c), X) \cong X(c)$*, naturally in each* $c : C$ *and* $X : \mathrm{PSh}(C)$*.*

Theorem (Descent for presheaf categories)

Let $E :\equiv \mathrm{PSh}(A)$ *and* $F :_\flat C \to E$*, then* $E / \varinjlim_{c:C} F(c) \simeq \varprojlim_{c:C} E/F(c)$*.*

Corollary (Internal natural models)

We can give an internal treatment of models of HoTT à la Awodey's natural models.

Kan extensions

Definition (Kan extensions)

Given $f :_\flat C \to D$ and a category E, the left Kan extension lan_f is the left adjoint to $f^* : E^D \to E^C$.

Theorem (Colimit formula)

If E is cocomplete, then lan_f exists. For $X :_\flat C \to E$ it computes to
$\mathrm{lan}_f X\, d \simeq \varinjlim(C \times_D D/d \to C \to E)$

Theorem (Quillen's Theorem A)

A functor $f :_\flat C \to D$ is right cofinal if and only if $L_{\mathbb{I}}(C \times_D d/D) \simeq \mathbf{1}$ for each $d :_\flat D$.

Corollary (Properness)

Cartesian fibrations are proper functors.

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