Completions in an (∞, n)-categorical setting

Miika Tuominen

University of Virginia

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Complete Segal spaces as $(\infty, 1)$-categories

Definition

A functor $W\colon \Delta^{\mathrm{op}} \to s\mathcal{S}et$ is a *Segal space* if

- it is injective/Reedy fibrant, and
- it satisfies the Segal condition: the Segal maps $W_m \to W_1 \underset{W_0}{\times} \cdots \underset{W_0}{\times} W_1$ are weak equivalences of spaces for each m.

A Segal space W is *complete* if each equivalence in W is homotopic to the identity on some object.

Rezk develops a homotopy theory of $(\infty, 1)$-categories as complete Segal spaces:

Theorem (Rezk)

There is a well-behaved model structure $\mathcal{CSS}$ on $\mathrm{Fun}(\Delta^{op}, s\mathcal{S}et)$ where fibrant objects are precisely complete Segal spaces.

Equivalences of $(\infty, 1)$-categories

Definition

A map $f \colon X \to Y$ is a *DK-equivalence* of $(\infty, 1)$-categories if

- it is surjective up to equivalence on objects, and
- it induces weak equivalences of spaces $\underline{\mathrm{Hom}}_X(a, b) \to \underline{\mathrm{Hom}}_Y(fa, fb)$ for all objects a and b of X.

Weak equivalences of complete Segal spaces behave like weak equivalences of $s\mathcal{S}et$-enriched categories:

Theorem (Rezk)

A map between complete Segal spaces is a DK-equivalence if and only if it is a weak equivalence in $\mathcal{CSS}$.

Completion of Segal spaces

DK-equivalences are also meaningful between Segal spaces without completeness.

Theorem (Rezk)

There is a fibrant replacement functor that takes each Segal space to a complete Segal space via a DK-equivalence.

Theorem (Rezk)

A map between Segal spaces is a DK-equivalence if and only if it is a weak equivalence in $\mathcal{CSS}$.

As a consequence, completeness may be understood as locality with respect to DK-equivalences.

Complete Segal Θ_n-spaces as (∞, n)-categories

To obtain a model of (∞, n)-categories, we may replace Δ with its n-categorical analogue, Θ_n.

Definition

A functor $W\colon \Theta_n^{\mathrm{op}} \to s\mathcal{S}et$ is a *Segal Θ_n-space* if

- it is injective/Reedy fibrant, and
- it satisfies a Segal condition for cells of all dimensions up to n.

A Segal Θ_n-space W is *complete* if each k-equivalence in W is homotopic to the identity on some $(k-1)$-cell for each k.

Theorem (Rezk)

There is a well-behaved model structure $\Theta_n Sp$ on $\mathrm{Fun}(\Theta_n^{op}, s\mathcal{S}et)$ where fibrant objects are precisely complete Segal Θ_n-spaces.

Equivalences of (∞, n)-categories

We extend the notion of DK-equivalence inductively:

Definition

A map $f\colon X \to Y$ is a *DK-equivalence* of (∞, n)-categories if

- it is surjective up to equivalence on objects, and
- it induces DK-equivalences of $(\infty, n-1)$-categories $\underline{\mathrm{Hom}}_X(a, b) \to \underline{\mathrm{Hom}}_Y(fa, fb)$ for all objects a and b of X.

Theorem (Bergner)

A map between complete Segal Θ_n-spaces is a DK-equivalence if and only if it is a weak equivalence in $\Theta_n Sp$.

Completions of Segal Θ_n-spaces

We extend Rezk's completion construction to each of the completeness conditions for Segal Θ_n-spaces.

Theorem (T.)

There is a fibrant replacement functor that takes each Segal Θ_n-space to a complete Segal Θ_n-space via a DK-equivalence.

We also generalize the characterization of DK-equivalences:

Theorem (T.)

A map between Segal Θ_n-spaces is a DK-equivalence if and only if it is a weak equivalence in $\Theta_n Sp$.