Triposes and toposes via arrow algebras

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Localic toposes and realizability toposes

topos $\mathrm{Sh}(X)$

$\uparrow$

frame X $(\leq\ \wedge\ \bigvee)$

Localic toposes and realizability toposes

topos $\mathsf{Set}[X^-]$

$\uparrow$

tripos $X^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

$\uparrow$

frame X $(\leq \wedge \bigvee)$

Higgs, *A category approach to boolean-valued set theory*, 1973

Fourman, Scott, *Sheaves and logic*, 1979

Hyland, Johnstone, Pitts, *Tripos theory*, 1980

Localic toposes and realizability toposes

topos $\mathsf{Set}[X^-]$

tripos $X^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

frame X $(\leq \; \wedge \; \bigvee)$

pca P $(\leq \; \cdot \; P^{\#})$

Hyland, Johnstone, Pitts, *Tripos theory*, 1980

Localic toposes and realizability toposes

topos $\mathsf{Set}[X^-]$

$\uparrow$

tripos $X^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

$\uparrow$

frame $X \quad (\leq \ \wedge \ \bigvee)$

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tripos $DP^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

$\uparrow$

pca $P \quad (\leq \ \cdot \ P^{\#})$

Hyland, Johnstone, Pitts, *Tripos theory*, 1980

Localic toposes and realizability toposes

topos $\mathsf{Set}[X^-]$

$\uparrow$

tripos $X^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

$\uparrow$

frame X $(\leq\ \wedge\ \bigvee)$

topos $\mathsf{Set}[DP^-]$

$\uparrow$

tripos $DP^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

$\uparrow$

pca P $\left(\leq\ \cdot\ P^{\#}\right)$

Hyland, Johnstone, Pitts, *Tripos theory*, 1980

Localic toposes and realizability toposes

topos $\mathsf{Set}[X^-]$	topos $\mathsf{Set}[DP^-]$
$\uparrow$	$\uparrow$
tripos $X^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$	tripos $DP^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$
$\uparrow$	$\uparrow$
frame $X \quad (\leq \ \wedge \ \bigvee)$	pca $P \quad (\leq \ \cdot \ P^{\#})$

Question

Can we find a common framework to study both localic and realizability toposes from a concrete, “elementary” level?

Arrow algebras

Arrow algebras are a generalization of Alexandre Miquel's implicative algebras aimed to factor through the two constructions.

Briët, van den Berg, *Arrow algebras*, 2023

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arrow algebra $A \;\; (\leq \; \bigwedge \; \to \; S)$

Briët, van den Berg, *Arrow algebras*, 2023

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tripos $A^{-} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre}$

$\uparrow$

arrow algebra A $(\leq \bigwedge \to S)$

Briët, van den Berg, *Arrow algebras*, 2023

Arrow algebras

Arrow algebras are a generalization of Alexandre Miquel's implicative algebras aimed to factor through the two constructions.

$$
\begin{array}{ll}
\text{topos} & \mathsf{Set}[A^-] \\
& \uparrow \\
\text{tripos} & A^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre} \\
& \uparrow \\
\text{arrow algebra} & A \quad (\leq \ \bigwedge \ \to \ S)
\end{array}
$$

Briët, van den Berg, *Arrow algebras*, 2023

Morphisms of arrow algebras

Morphisms between arrow algebras can be defined as functions "preserving" the implication $\to$ and the subset S.

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

Morphisms between arrow algebras can be defined as functions "preserving" the implication $\to$ and the subset S.

More precisely, a morphism $f : (A, \leq, \to, S_A) \to (B, \leq, \to, S_B)$ is a function $f : A \to B$ such that:

1. $f(S_A) \subseteq S_B$;
2. $$\bigwedge_{a,a' \in A} f(a \to a') \to f(a) \to f(a') \in S_B;$$
3. for every $I \subseteq A \times A$,
$$\text{if} \bigwedge_{(a,a') \in I} a \to a' \in S_A \text{ then } \bigwedge_{(a,a') \in I} f(a) \to f(a') \in S_B.$$

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

These morphisms specialize to:

- finite-meets preserving maps of frames;
- partial applicative morphisms of pcas,

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Morphisms of arrow algebras

These morphisms specialize to:

- finite-meets preserving maps of frames;
- partial applicative morphisms of pcas,

and correspond exactly to:

- cartesian transformations of the induced triposes.

$$\mathsf{ArrAlg} \qquad\qquad \mathsf{Tripos}_{\mathsf{cart}}$$

$$A \xrightarrow{\quad f \quad} B \qquad \simeq \qquad A^- \xrightarrow{\quad f \circ - \quad} B^-$$

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

In particular, morphisms which are left adjoints in the preorder-enriched category ArrAlg specialize to:

- homomorphisms of frames;
- computationally dense partial applicative morphisms of pcas,

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

In particular, morphisms which are left adjoints in the preorder-enriched category ArrAlg specialize to:

- homomorphisms of frames;
- computationally dense partial applicative morphisms of pcas,

and correspond exactly to:

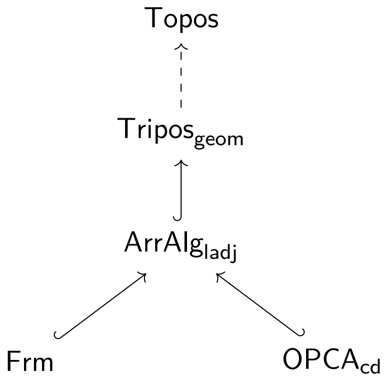
- geometric transformations of the induced triposes.

$$\mathsf{ArrAlg}_{\mathsf{ladj}} \qquad\qquad\qquad \mathsf{Tripos}_{\mathsf{geom}}$$

$$A \underset{h}{\overset{f}{\rightleftarrows}} B \;\;(f \dashv h) \qquad \simeq \qquad A^{-} \underset{h \circ -}{\overset{f \circ -}{\rightleftarrows}} B^{-} \;\;(f\circ - \dashv h \circ -)$$

T., *A category of arrow algebras for modified realizability*, 2024

Categorically, we have the following picture:



Subtoposes and nuclei

As an example of an application, we can characterize subtoposes in terms of nuclei on the underlying arrow algebra, generalizing what happens for locales.

Proposition

Let A be an arrow algebra. Then, every subtopos of $\mathsf{Set}[A^-]$ is induced by a geometric transformation of triposes

$$A^- \underset{j \circ -}{\overset{\mathrm{id}_A \circ -}{\underset{\perp}{\rightleftarrows}}} A_j^-$$

for some nucleus j on A.

Thank you!