Computation up to Isomorphism: Higher-Dimensional Realizability

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Groupoidal combinatory algebras

- A GCA consists of a groupoid A and an "application" functor:

$$(-) \bullet (=) : A \times A \to A$$

  such that there exist "combinators" $\mathsf{s}, \mathsf{k} \in A$ and natural isomorphisms:

$$\sigma_{a,b,c} : \mathsf{s}abc \to ac(bc)$$
$$\kappa_{a,b} : \mathsf{k}ab \to a$$

- Combinatory completeness: for every polynomial $t(x_1, ..., x_n)$ over A there exists $b \in A$ and a natural isomorphism:

$$\beta_{\vec{a}} : ba_1...a_n \to t(a_1, ..., a_n)$$

Examples of GCAs

- 2-dimensional models of the λ-calculus, ie. cartesian closed bicategories $\mathcal{B}$ with a pseudoreflexive object $U \in \mathcal{B}$.
- Instances include: generalised species of structures[1], profunctorial Scott semantics[2], a categorified relational model[3] and a categorified graph model[4].
- The carrier groupoid is Core$(\mathcal{B}(1, U))$ and the application functor is induced by:

$$U \times U \xrightarrow{U \times r_{U^U}} U \times U^U \xrightarrow{\text{eval}} U$$

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[1] Fiore, Gambino, Hyland, and Winskel, "The cartesian closed bicategory of generalised species of structures", 2008.

[2] Galal, "A profunctorial Scott semantics", 2020.

[3] Olimpieri, "Intersection type distributors", 2021.

[4] Kerinec, Manzonetto, and Olimpieri, "Why are proofs relevant in proof-relevant models?", 2023.

Groupoidal Assemblies over a GCA A

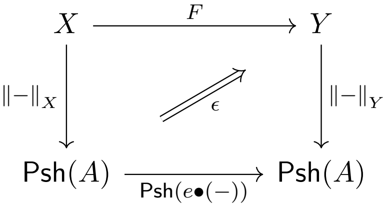
- A groupoidal assembly consists of a groupoid X and a "realizability" functor:

$$\|-\|_X : X \to \mathsf{Psh}(A)$$

  such that for every $x \in X$ there is some $a \in A$ such that $\|x\|_X(a) \neq \varnothing$.
- Proof-relevant realizability: think of $t \in \|x\|_X(a)$ as a proof that a realizes x; write $a \Vdash_X t : x$.

Morphisms of groupoidal assemblies

- A morphism $(X, \|-\|_X) \to (Y, \|-\|_Y)$ of groupoidal assemblies over A is a functor $F : X \to Y$ such that there exists $e \in A$ and a natural transformation:



  where $\mathsf{Psh}(J) \dashv (-) \circ J$. Write $e \Vdash \epsilon : F$.

- 2-cells are natural transformations.

The (2,1)-category $\mathbf{GA}$...

- is finitely complete:

$$\mathbf{GA}(X, Y^{\nabla I}) \cong \mathbf{Gpd}(I, \mathbf{GA}(X, Y))$$
$$\nabla : \mathbf{Gpd} \to \mathbf{GA}$$
$$Z \mapsto (Z, z \mapsto 1)$$

- is cartesian closed.

Outlook

- $\mathbf{GA}$ should be regular and locally cartesian closed.
- Investigate the ex/reg completion of $\mathbf{GA}$—is this a non-Grothendieck (2,1)-topos?
- Impredicative universe of 1-types: modest groupoids and inverse semicateads?[5]

$$\mu_{x,x'} : \left[\int \|x\|_X \times_A \int \left\|x'\right\|_X\right]_0 \cong X(x, x')$$

[5] Bourn and Penon, "2-catégories réductibles", 1978.