A study of Kock's fat Delta

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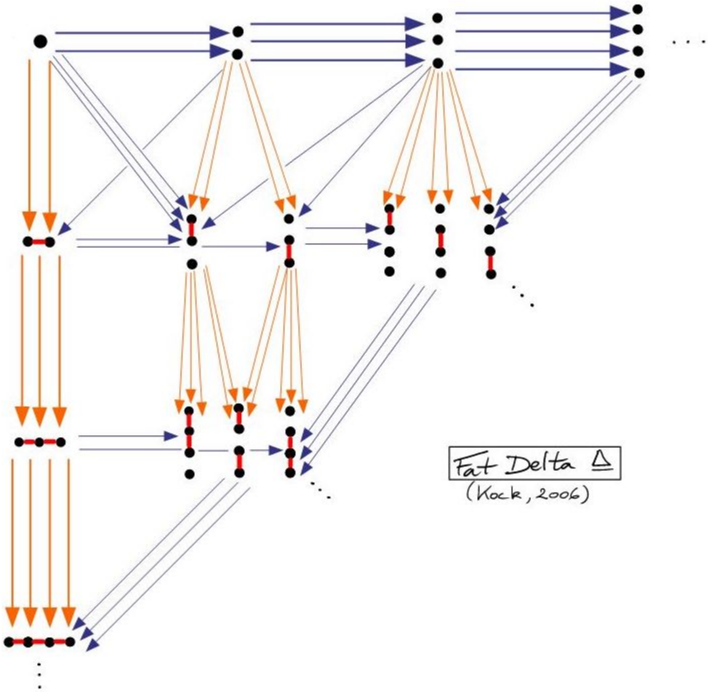
Fat Delta Δ
(Kock, 2006)

Introduction of $\underline{\Delta}$

Motivation: Simpson's conjecture

- Motivation for Kock (2006) to introduce $\underline{\Delta}$:
  - The simplex category Δ with **degeneracies up to homotopy**
  - The identity coherence structure is part of the data as objects
- Motivation, in low dimension, for Paoli (2024) to study $\underline{\Delta}$ further

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Motivation: higher categories in homotopical type theories

- Construct Reedy fibrant diagrams over direct categories
- Use simplicial methods

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Definition (Fat Delta $\underline{\Delta}$, Kock 2006)

The category $\underline{\Delta}$ is the category of **relative finite non-empty semiordinals** and maps between them.

Monad with arities

Study $\underline{\Delta}$ using the theory of **monads with arities**.

- Generalised nerve constructions
- Algebras as presheaves with (Segal) conditions
- Correspondence: monads $\longleftrightarrow$ algebraic theories

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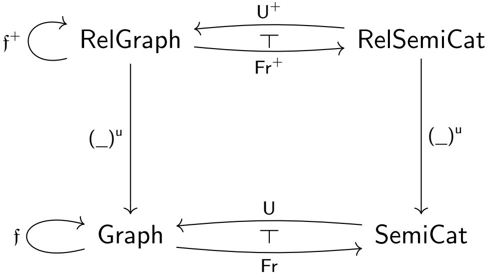
We rather work with a sufficient property:

Theorem (Berger, Melliès, and Weber 2012)

Suppose T is a **strongly cartesian monad** *(cartesian + local right adjoint) on a (finitely complete) category $\mathcal{E}$ with a dense generator, then T is a* **monad with arities***.*

The free relative semicategory monad $\mathfrak{f}^+$

Using forgetful functors we can construct the following diagram:



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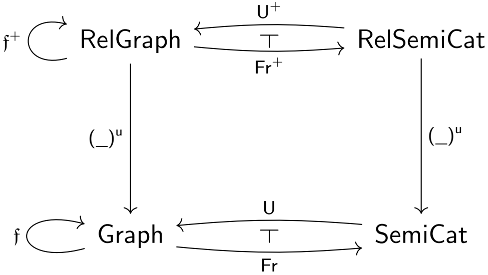

- $\mathfrak{f}\, G$ = paths $[n] \to G$ and concatenation

Proposition

The **free semicategory** *monad* $\mathfrak{f} : \mathsf{Graph} \to \mathsf{Graph}$ *is strongly cartesian.*

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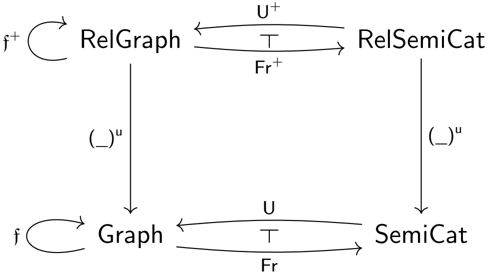

- $\mathfrak{f}\,G$ = paths $[n] \to G$ and concatenation
- $\mathfrak{f}^+(G^+ \hookrightarrow G) = \mathfrak{f}\,G^+ \hookrightarrow \mathfrak{f}\,G$

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Proposition

The **free relative semicategory** *monad* $\mathfrak{f}^+ : \mathsf{RelGraph} \to \mathsf{RelGraph}$ *is strongly cartesian.*

Results

Theorem (Arities $\underline{\Delta}_0$ and the associated theory $\underline{\Delta}$)

- *The category of arities $\underline{\Delta}_0$ can be identified with the subcategory of $\underline{\Delta}$ consisting of relative semiordinals and* **distance-preserving morphisms**.
- *The associated theory is the category of free $\mathfrak{f}^+$-algebras over $\underline{\Delta}_0$, that is the category $\underline{\Delta}$ of* **relative semiordinals**.

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Theorem (Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$)

The category $\underline{\Delta}$ has an **active-inert** *factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$ consisting of distance-preserving and endpoint-preserving morphisms.*

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Theorem (Nerve theorem for $\underline{\mathcal{N}}$)

The nerve functor $\underline{\mathcal{N}} : \mathsf{RelSemiCat} \to \widehat{\underline{\Delta}}$ is **fully faithful**, *and the essential image is spanned by the presheaves whose restriction along $\underline{\Delta}_0 \hookrightarrow \underline{\Delta}$ belongs to the essential image of $\underline{\mathcal{N}} : \mathsf{RelGraph} \to \widehat{\underline{\Delta}_0}$.*

Conclusion

Contribution

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- Active-inert factorisation system $(\underline{\Delta}_a, \underline{\Delta}_0)$
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Work in progress

- $\underline{\Delta}$ as a hypermoment category (in the sense of Berger (2022))
- $\underline{\Delta}$-spaces

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For more see: https://www.stiephenpradal.com

Thank you!

Bibliography

Berger, Clemens (2022). "Moment Categories and Operads". In: *Theory and Application of Categories* 38.9. URL: http://www.tac.mta.ca/tac/volumes/38/39/38-39.pdf.

Berger, Clemens, Paul-André Melliès, and Mark Weber (2012). "Monads with Arities and Their Associated Theories". In: *Journal of Pure and Applied Algebra* 216.8. DOI: 10.1016/j.jpaa.2012.02.039.

Kock, Joachim (2006). "Weak Identity Arrows in Higher Categories". In: *International Mathematics Research Papers*. DOI: 10.1155/IMRP/2006/69163.

Kraus, Nicolai and Christian Sattler (2017). *Space-Valued Diagrams, Type-Theoretically (Extended Abstract)*. DOI: 10.48550/arXiv.1704.04543.

Paoli, Simona (2024). *Weakly Globular Double Categories and Weak Units*. DOI: 10.48550/arXiv.2008.11180.