Sketchable infinity categories

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Sketches

A **sketch** Σ is a triple (A, L, C) of a small category A, a set L of cones and a set C of cocones. If $C = \emptyset$, Σ is a **limit sketch**.

Example. Let A be the small category generated by the square α:

$$\begin{array}{ccc} & A & \\ 0 & \longrightarrow & 1 \\ \downarrow & \alpha & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

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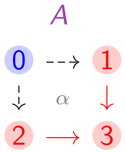
Observe that α is a cone with apex and diagram

Then $(A, \{\alpha\}, \emptyset)$ is a sketch



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A **model** of a sketch in a category K is a functor $F : A \to K$ which sends cones of L to limits and cocones of C to colimits.

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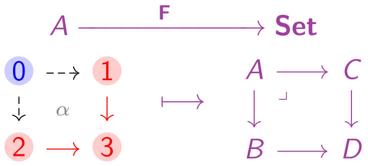

A model **F** of the sketch $(A, \{\alpha\}, \emptyset)$ is a pullback of sets

Representation theorem

$$\textbf{Limit-sketchable} \iff \text{Equivalent to } \mathrm{Mod}(\Sigma) \text{ for a limit sketch } \Sigma$$
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A category is presentable if and only if it is limit-sketchable.

Theorem (Lair 1981)

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Goal

Generalize these results to ∞-categories

Higher sketches (Joyal 2008)

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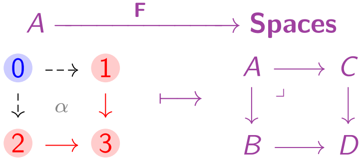
Example. Let $\mathcal{A}$ be the nerve of the category generated by α:

Observe that α is a cone with apex and diagram

Then $(\mathcal{A}, \{\alpha\}, \emptyset)$ is a sketch



A model **F** of the sketch $(\mathcal{A}, \{\alpha\}, \emptyset)$ is a homotopy pullback

Examples

- **Algebraic theories** (Rosicky 2007): Monoid objects (A_∞-spaces), commutative monoid objects (E_∞-spaces), group objects (∞-groups), R-modules, ...

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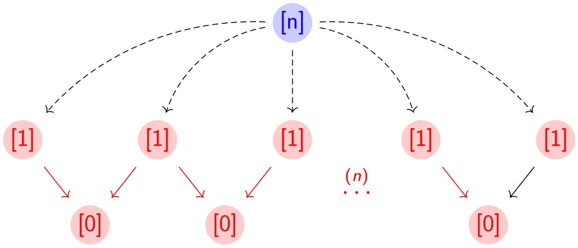
- Spectrum object (Spectra)
- Univalent category objects (Complete Segal spaces) and univalent operad objects (Complete Segal dendroidal spaces)
- ∞-sheaves $\simeq$ hypercomplete ∞-topos

Examples: Category objects

Let $\mathcal{K}$ be a complete ∞-category, and α_n be the cone over Δ^{op} with apex and diagram for all $n \in \mathbb{N}$:



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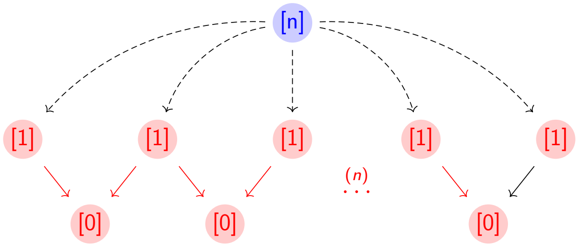

Then $\Sigma = (\Delta^{\mathrm{op}}, \{\alpha_n \mid n \in \mathbb{N}\})$ is a limit sketch, and a model $F : \Delta^{\mathrm{op}} \to \mathcal{K}$ is a simplicial object in $\mathcal{K}$ such that

$$F_n \xrightarrow{\sim} F_1 \times_{F_0} F_1 \times_{F_0} \overset{(n)}{\cdots} \times_{F_0} F_1. \qquad \textit{(Segal condition)}$$

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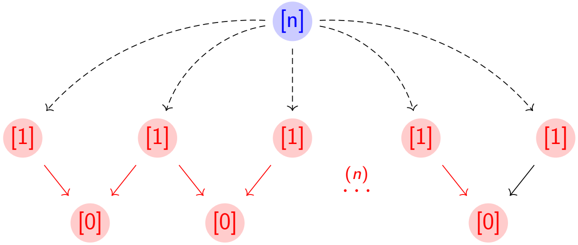

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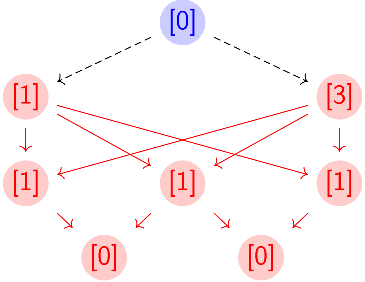
$\mathrm{Mod}(\Sigma, \mathcal{K}) \simeq$ **Category objects on** $\mathcal{K}$

$\mathrm{Mod}(\Sigma) \simeq$ **Segal spaces**

Examples: Univalent category objects

Let $\mathcal{L}_S$ be the set of cones of the previous example, and β be the cone over Δ^{op} with apex and diagram:



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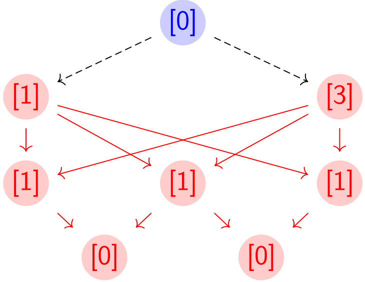

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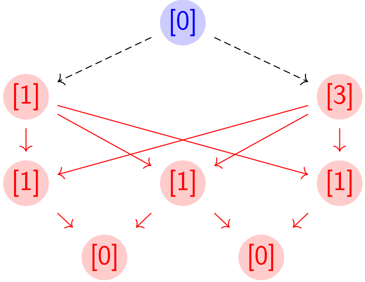

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$\mathrm{Mod}(\Sigma, \mathcal{K}) \simeq$ **Univalent category objects on** $\mathcal{K}$

$\mathrm{Mod}(\Sigma) \simeq$ **Complete Segal spaces**

Higher representation theorem

Theorem. An ∞-category is:

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- ✓ Presentability definitions of Joyal and Lurie are equivalent.
- ✓ Can be used to prove that many ∞-categories are presentable.
- ✓ The result can be extended to models of sketches over any presentable ∞-category.
- ? Example of a sketch with ∞-category of models that is accessible but neither presentable nor 1-categorical.

Thank you for listening!

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- Chu, Hongyi and Rune Haugseng (2021). "Homotopy-coherent algebra via Segal conditions". *Advances in Mathematics* 385.

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