Pure maps are strict monomorphisms

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The proofs are written down in: arxiv.org/abs/2407.13448

Theorem (Lurie)

$\mathcal{C}$ is coherent with fin. disjoint coproducts. Then every $\mathcal{C} \to \mathbf{Set}$ regular functor factors uniquely as $\mathcal{C} \xrightarrow{M} Sh(B) \xrightarrow{\Gamma} \mathbf{Set}$, where M is coherent and B is a Boolean-alg. with the finite union topology.

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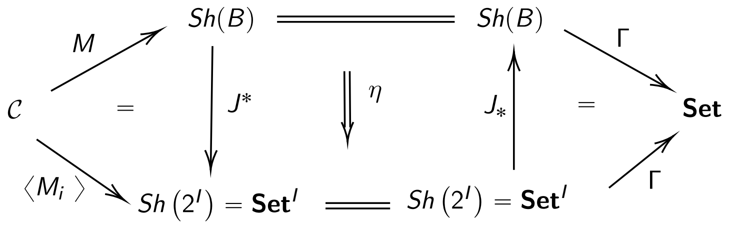

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5. *reg* $\hookrightarrow \prod$ *coh*: by Lurie's thm.

References

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Pure maps are strict monomorphisms (arxiv.org/abs/2407.13448)