A Higher-Dimensional Eckmann–Hilton Argument

Eugenia Cheng | Alex Corner

School of the Art Institute Chicago | Sheffield Hallam University

Extended Slides: alex-corner.github.io/slides/corner-ct24.pdf

The Concept of Degeneracy

category		monoid
0-cells } trivial		-
1-cells	→	elements
composition	→	multiplication
identity	→	unit

The Concept of Degeneracy

category		monoid
0-cells } trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category	k-d-n-category
0-cells	
1-cells	
$\vdots$	
$(k-1)$-cells	
k-cells	
$\vdots$	
$(n-1)$-cells	
n-cells	

The Concept of Degeneracy

category		monoid
0-cells } trivial		-
1-cells	$\longrightarrow$	elements
composition	$\longrightarrow$	multiplication
identity	$\longrightarrow$	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells	trivial	-
1-cells	trivial	-
$\vdots$	trivial	$\vdots$
$(k-1)$-cells	trivial	-
k-cells	$\longrightarrow$	0-cells
$\vdots$	$\vdots$	$\vdots$
$(n-1)$-cells	$\longrightarrow$	$(n-k-1)$-cells
n-cells	$\longrightarrow$	$(n-k)$-cells

Idea: A k-degenerate n-category 'is' an $(n-k)$-category with extra structure.

The Concept of Degeneracy

category		monoid
0-cells} trivial		-
1-cells	⟶	elements
composition	⟶	multiplication
identity	⟶	unit

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

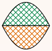
n-category		k-d-n-category
0-cells (trivial)		-
1-cells (trivial)		-
⋮ (trivial)		⋮
$(k-1)$-cells (trivial)		-
k-cells	⟶	0-cells
⋮	⋮	⋮
$(n-1)$-cells	⟶	$(n-k-1)$-cells
n-cells	⟶	$(n-k)$-cells

Idea: A k-degenerate n-category 'is' an $(n-k)$-category with extra structure.

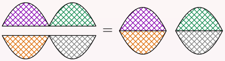
1-degenerate 2-categories

2-category		monoidal category
0-cells} trivial		-
1-cells	⟶	objects
2-cells	⟶	morphisms
hor. composites	⟶	$\otimes$

vert. composites	⟶	composition



identity	⟶	unit object
interchange	⟶	interchange



$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$$

2-degenerate 2-categories

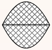
2-category		comm. monoid
0-cells (trivial)		-
1-cells (trivial)		-
2-cells	⟶	elements
hor. composites	⟶	multiplication $*$
vert. composites	⟶	multiplication $\circ$
identity	⟶	unit for $*$ and $\circ$
interchange	⟶	interchange

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

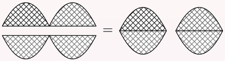
1-degenerate 2-categories

2-category		monoidal category
0-cells } trivial		-
1-cells	⟶	objects
2-cells	⟶	morphisms
hor. composites	⟶	$\otimes$

vert. composites	⟶	composition



identity	⟶	unit object
interchange	⟶	interchange



$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$$

2-degenerate 2-categories

2-category		comm. monoid
0-cells	trivial	-
1-cells	trivial	-
2-cells	$\longrightarrow$	elements
hor. composites	$\longrightarrow$	multiplication $*$
vert. composites	$\longrightarrow$	multiplication $\circ$
identity	$\longrightarrow$	unit for $*$ and $\circ$
interchange	$\longrightarrow$	interchange

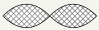
$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

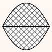
Idea: $\circ$ and $*$ are the same and commutative.
This is the Eckmann–Hilton argument:

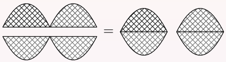
$$\begin{aligned} a \circ b &= (1 * a) \circ (b * 1) \\ &= (1 \circ b) * (a \circ 1) \\ &= b * a \\ &= (b \circ 1) * (1 \circ a) \\ &= (b * 1) \circ (1 * a) \\ &= b \circ a \end{aligned}$$

1-degenerate 2-categories

2-category		monoidal category
0-cells	trivial	-
1-cells	$\longrightarrow$	objects
2-cells	$\longrightarrow$	morphisms
hor. composites	$\longrightarrow$	$\otimes$
vert. composites	$\longrightarrow$	composition
identity	$\longrightarrow$	unit object
interchange	$\longrightarrow$	interchange

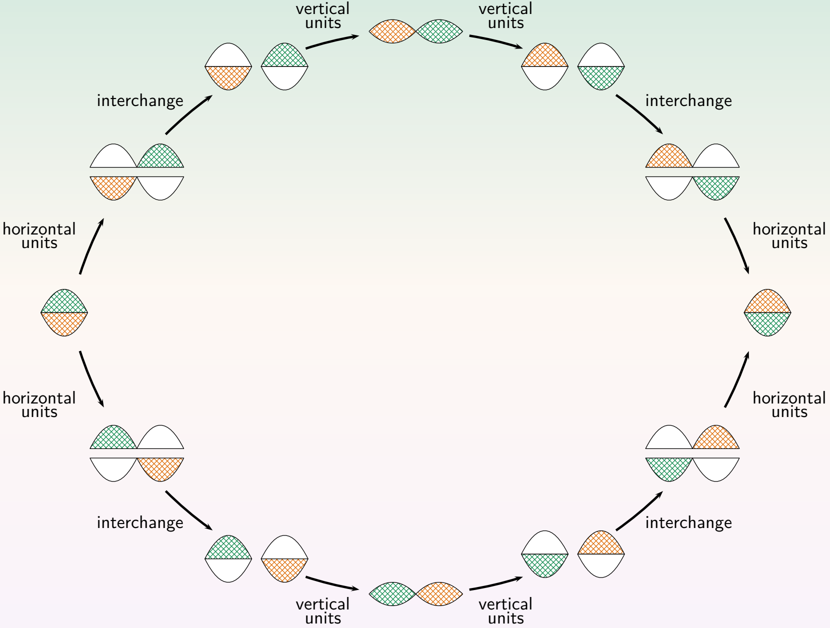






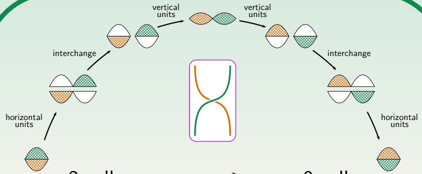
$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$$

Eckmann–Hilton Argument



2-degenerate 3-categories

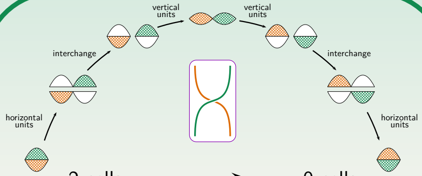
2-degenerate 3-categories



2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	$\otimes$s
interchange	⟶	braiding

braided monoidal categories

2-degenerate 3-categories



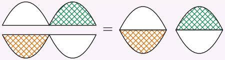
2-cells	⟶	0-cells
3-cells	⟶	1-cells
composites	⟶	$\otimes$s
interchange	⟶	braiding

E–H

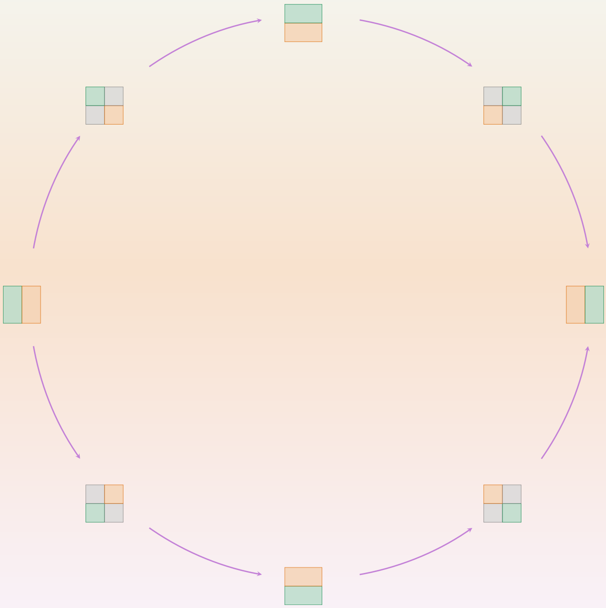
braided monoidal categories

Coherence in 3 flavours

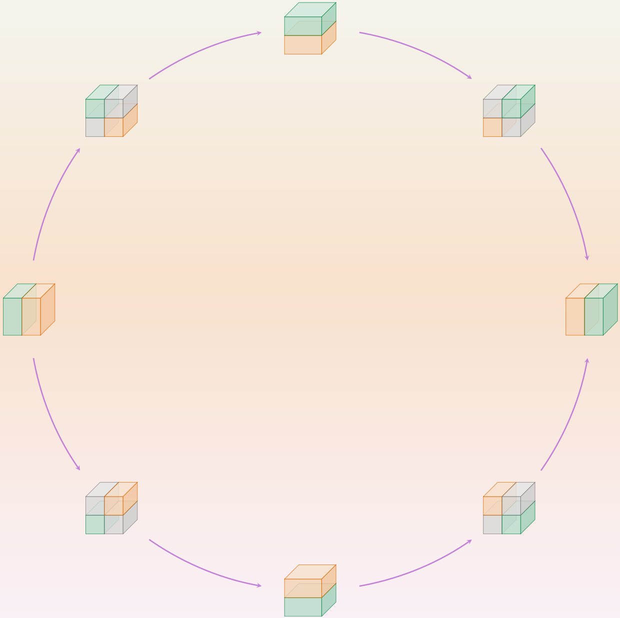
	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



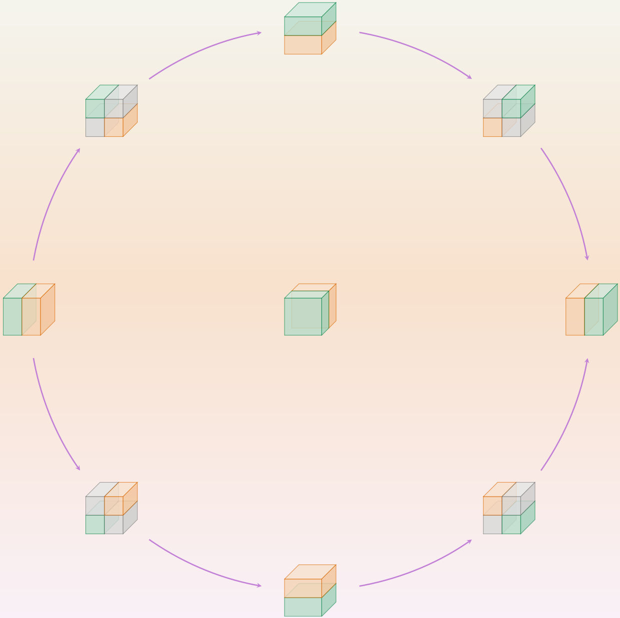
The Eckmann–Hilton Clock



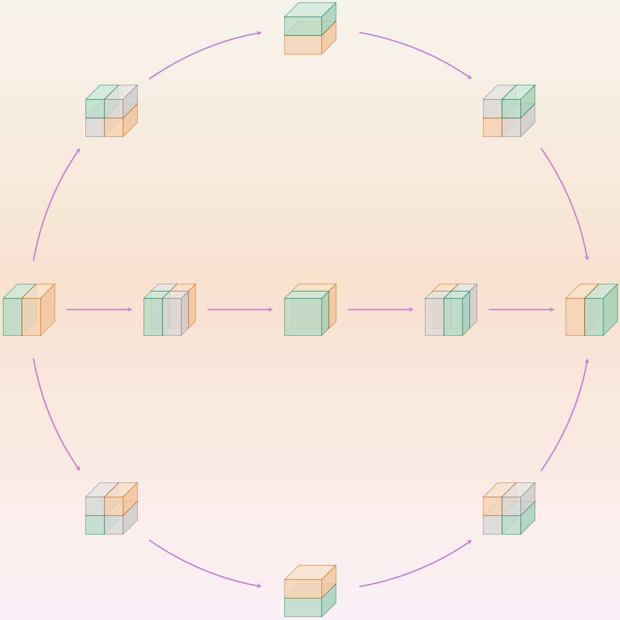
The Eckmann–Hilton Sphere



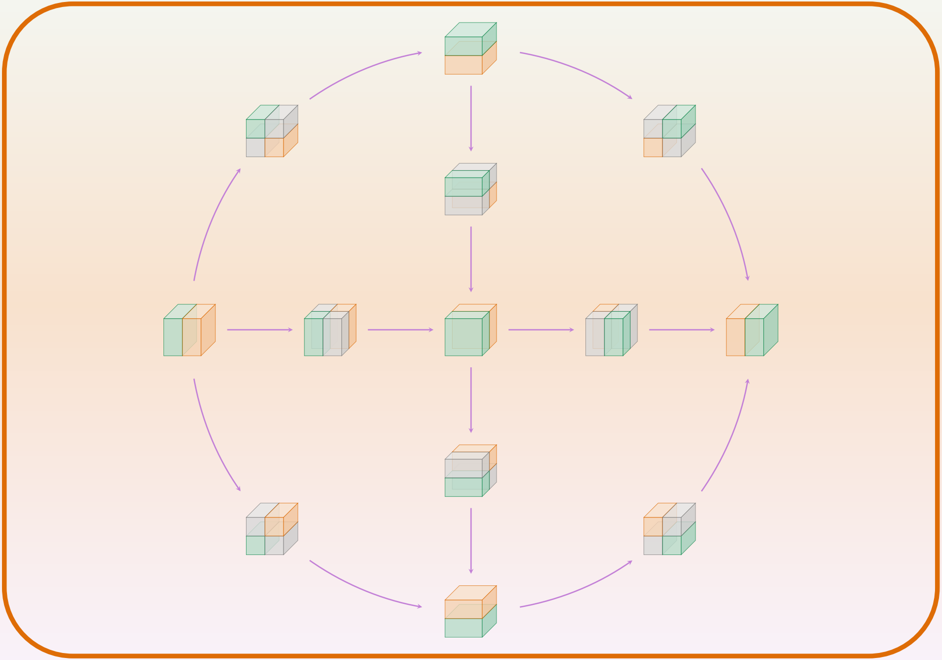
The Eckmann–Hilton Sphere



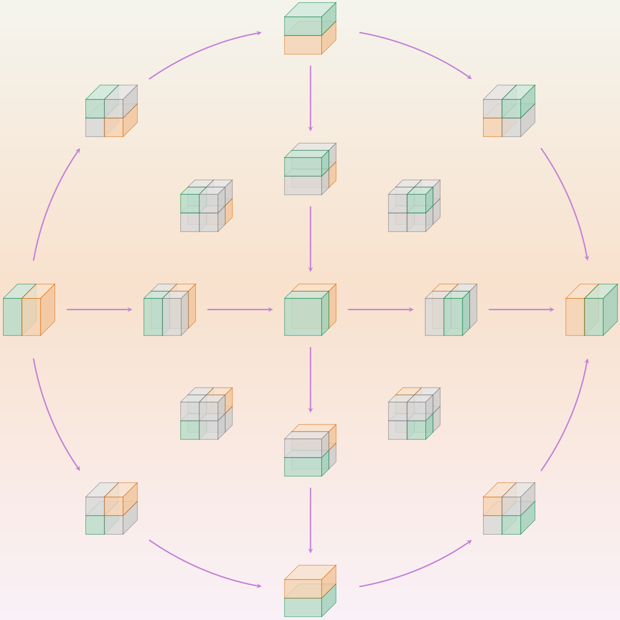
The Eckmann–Hilton Sphere



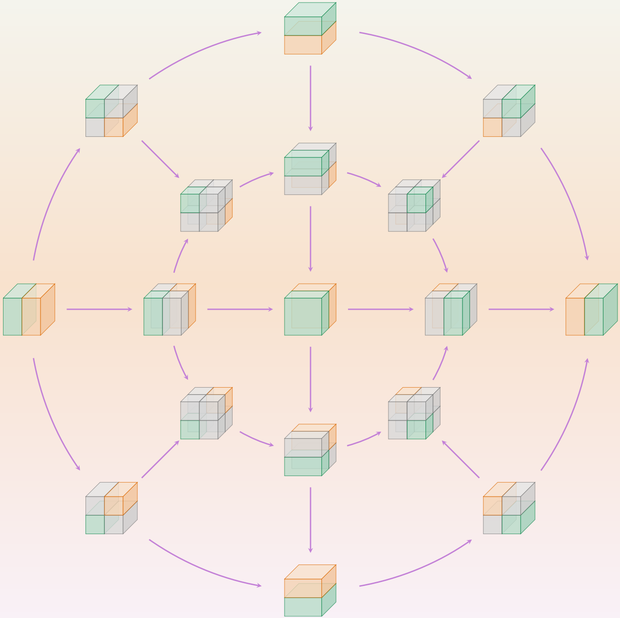
The Eckmann–Hilton Sphere



The Eckmann–Hilton Sphere



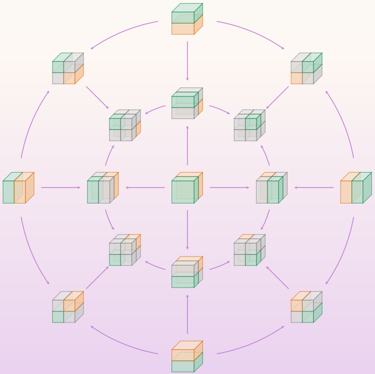
The Eckmann–Hilton Sphere



Summary

We have a heirarchy of results in progress demonstrating that all of the following produce symmetric monoidal categories:

- 3-tuply monoidal categories: two weak, one strict, strict interchanges
- 3-degenerate 4-categories: produced using iconic constructions
- $(n-1)$-degenerate n-categories: produced using iconic constructions



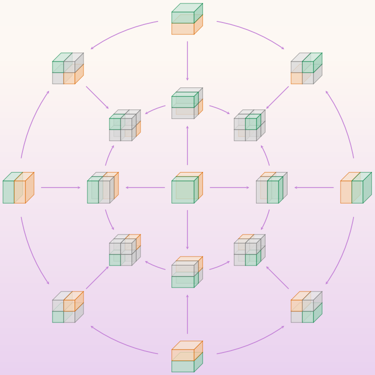
Summary

We have a heirarchy of results in progress demonstrating that all of the following produce symmetric monoidal categories:

- 3-tuply monoidal categories: two weak, one strict, strict interchanges
- 3-degenerate 4-categories: produced using iconic constructions
- $(n-1)$-degenerate n-categories: produced using iconic constructions

Future Work

- Totalities: Do this for triply-degenerate 4-categories and $(n-1)$-degenerate n-categories.
- Combinatorics: Investigate the interesting structures arising from weak interchange and the Eckmann–Hilton sphere.
- Higher Dimensions: Look at the higher Eckmann–Hilton spheres.



Thank you!