Homotopy Languages

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“Evil” properties

Theorem (Blanc, Freyd)

An elementary property on categories is invariant under equivalence of categories if it is a diagrammatic property.

Take the following sentence in context x, y, w objects and $f \in \mathrm{hom}(x, y), k, l \in \mathrm{hom}(y, w)$:

$$Eq_{k,l}(f) := kf = lf, \forall z \in \mathrm{Ob}, \forall g \in \mathrm{hom}(z, y), kg = lg$$
$$\exists h \in \mathrm{hom}(z, x), fh = g, \forall h \in \mathrm{hom}(z, x), fh' = g, h = h'$$

Key point: Such sentences do not include the equality between objects.

Inspired from Makkai’s work, we obtain a comparable result for model categories.

Summary of the work

- Given a (weak) model category we build a first order infinitary logic. The formulas in this language express properties about fibrant objects. The cofibrant objects play the role of contexts, and maps from a cofibrant object to a fibrant object the role of variables.

Theorem

- **First invariance:** *Homotopic maps satisfy the same of formulas.*
- **Second invariance:** *Homotopically equivalent fibrant objects validate the same formulas.*
- **Third invariance:** *Homotopically equivalent cofibrant objects (contexts) give equivalent set of formulas.*
- **Fourth invariance:** *Two Quillen equivalent (weak) model categories have equivalent languages.*

Idea of the construction

$$\text{Cofibrations} \longmapsto \text{(Infinitary) Generalized algebraic theory}$$

The case for categories:

$$\emptyset \to \{*\} \longmapsto \vdash \mathrm{Ob}\ \mathrm{Type}$$

$$\{0\} \sqcup \{1\} \to \{0 \to 1\} \longmapsto x, y : \mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \mathrm{Type}$$

$$\{0 \rightrightarrows 1\} \to \{0 \to 1\} \longmapsto x, y : \mathrm{Ob}, f, g : \mathrm{Hom}(x, y) \vdash (f = g)\ \mathrm{Type}$$

This allows to get an explicit syntax from the model structure.

Further examples

- In the language bicategories or 2-categories. We are allowed to speak about equality between 2-cells, but not between 1- or 0-cells.
- The language associated to the projective model structure on chain complexes allows to talk about chains with a specified boundary. There is no equality between chains x, x', we only say $\partial y = x - x'$.
- In the language of the Joyal or the Kan-Quillen model on simplicial sets we can talk about simplicies which satisfy some boundary condition. The point that differentiate them is the fibrant objects in each model structure.

Thank you!