Enriched duality in double categories

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Workshop on Categorical Logic and Higher Categories
University of Manchester

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Outline

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The category Alg of k-algebras is enriched in Coalg of k-coalgebras.

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4. Further directions

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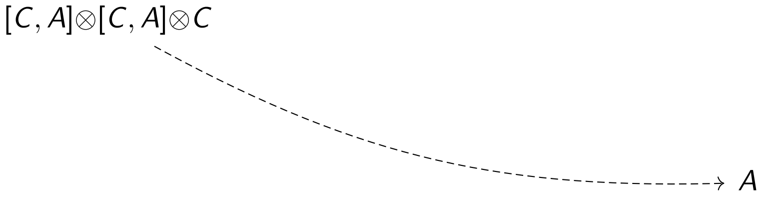

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More generally, there exists *universal measuring k-coalgebra* with $\mathrm{Alg}(A,\mathrm{Hom}_k(C,B))\cong\mathrm{Coalg}(C,P(A,B))$ – so $A^o=P(A,k)$.

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The category Mon is enriched in the monoidal Comon.

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> 0-cells, horizontal 1-cells, *globular* 2-maps
> make horizontal bicategory $\mathcal{H}(\mathbb{D})$.

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For $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ of sets, functions and $\mathcal{V}$-matrices $S\colon X \nrightarrow Y$ i.e. $\{S(x,y)\} \in \mathcal{V}$

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$$\begin{array}{ccccc} X & \overset{A}{\nrightarrow} & X & \overset{A}{\nrightarrow} & X \\ \| & & \Downarrow\mu & & \| \\ X & & \underset{A}{\nrightarrow} & & X \end{array} \qquad \begin{array}{ccc} X & \overset{1_X}{\nrightarrow} & X \\ \| & \Downarrow\eta & \| \\ X & \underset{A}{\nrightarrow} & X \end{array}$$

Dually, comonad $C\colon X \nrightarrow X$. These form categories $\mathsf{Mnd}(\mathbb{D})$ & $\mathsf{Cmd}(\mathbb{D})$.

$\star$ Morphisms are different than those for (co)monads in bicategories.

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$$
\begin{array}{ccccc}
X & \xrightarrow[\shortmid]{A} & X & \xrightarrow[\shortmid]{A} & X \\
\| & & \Downarrow\mu & & \| \\
X & & \xrightarrow[A]{\qquad\shortmid\qquad} & & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow[\shortmid]{1_X} & X \\
\| & \Downarrow\eta & \| \\
X & \xrightarrow[A]{\shortmid} & X
\end{array}
$$

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A $\mathcal{V}$-cocategory comes with cocomposition $C(x,z) \to \sum_y C(x,y) \otimes C(y,z)$ and coidentities $C(x,x) \to I$, coassociative and counital.

For $\mathbb{D}$ a (...) double category, $\mathsf{Mnd}(\mathbb{D})$ is enriched in $\mathsf{Cmd}(\mathbb{D})$.

For $\mathbb{D}$ a (...) double category, Mnd($\mathbb{D}$) is enriched in Cmd($\mathbb{D}$).

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$\mathsf{Mnd}(\mathbb{D})$ and $\mathsf{Cmd}(\mathbb{D})$ are monoidal, with $C \otimes D$ comonad via
$C \otimes D \to (C \circ C) \otimes (D \circ D) \cong (C \otimes D) \circ (C \otimes D)$.

Monoidal closed

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In $\mathcal{V}$-$\mathbb{M}$at, Set is l. p. & $\mathcal{V}$-$\mathbb{M}\mathrm{at}_1$ is too as pullback

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbb{M}\mathrm{at}_1 & \rightarrow & \mathrm{Fam}(\mathcal{V}) \\ {\scriptstyle (\mathfrak{s},\mathfrak{t})}\downarrow & & \downarrow \\ \mathrm{Set}^2 & \xrightarrow{\times} & \mathrm{Set} \end{array}$$

by the Limit Theorem.

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◇ Obtain enrichment of $\mathrm{Mnd}(\mathbb{D})$ in $\mathrm{Cmd}(\mathbb{D})$!

Moving to other contexts

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$\star$ For $\mathcal{V}$ symmetric monoidal closed & locally presentable, $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$ is a fibrant, monoidal closed & locally presentable double category.

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Goal: employ/*extend* theory to apply to $\mathbb{D} = \mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$ (operads)

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operads

- objects are sets $X, Y, \ldots$
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Horizontal composition is generalization of substitution for species.

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Horizontal composition is generalization of substitution for species.

■ $\mathcal{V}\text{-}\mathbb{S}\mathrm{ym}$ is fibrant

Moving to other contexts

⋆ For $\mathcal{V}$ symmetric monoidal closed & locally presentable, $\mathbb{D} = \mathcal{V}$-$\mathbb{M}$at is a fibrant, monoidal closed & locally presentable double category.

- Enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories

Goal: employ/*extend* theory to apply to $\mathbb{D} = \mathcal{V}$-$\mathbb{S}$ym

operads

- objects are sets $X, Y, \ldots$
- vertical 1-cells are functions $f, g, \ldots$
- horizontal 1-cells are coloured symmetric sequences $M\colon SX^{\mathrm{op}} \times Y \to \mathcal{V}$
- 2-maps are $M(x_1, \ldots, x_n; y) \to N(fx_1, \ldots, fx_n; gy)$

Horizontal composition is generalization of substitution for species.

■ $\mathcal{V}$-$\mathbb{S}$ym is fibrant; but not monoidal double anymore!

Oplax monoidal double structure

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- A double category is *oplax* monoidal with comparison maps

$$(N \circ M) \otimes (N' \circ M') \to (N \otimes N') \circ (M \otimes M'), \quad 1_X \otimes 1_{X'} \to 1_{X \otimes X'}$$
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$\star$ Induced tensor is $M \boxtimes N = X \otimes Z \overset{M \otimes N}{\nrightarrow} TY \otimes TW \overset{\hat{\tau}}{\nrightarrow} T(Y \otimes W)$.

Coloured symmetric sequences

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⋆ 'Free symmetric strict monoidal category 2-monad' $S\colon \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}$

$$\text{with } S_n(C)((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{\sigma} \prod_{1\leq i\leq n} C(x_{\sigma(i)}, y_i)$$

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- Oplax monoidal structure is many-object *arithmetic product* of species

$$(M \boxtimes N)(\vec{a},(x,z)) = \int^{\vec{y},\vec{w}} S(Y\times W)(\vec{a},\vec{y}\boxtimes\vec{w}) \times M(\vec{y},x) \times N(\vec{w},z)$$

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[One-object case] If $\mathcal{V}$ is symmetric monoidal closed and loc presentable,

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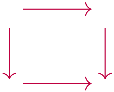
- Extend full story
$$\begin{array}{ccc} \mathsf{Mod}(\mathbb{D}) & \overset{\text{enriched}}{\dashrightarrow} & \mathsf{Comod}(\mathbb{D}) \\ {\scriptstyle\text{fibered}}\downarrow & & \downarrow{\scriptstyle\text{opfibered}} \\ \mathsf{Mnd}(\mathbb{D}) & \underset{\text{enriched}}{\dashrightarrow} & \mathsf{Cmd}(\mathbb{D}) \end{array}$$
to oplax monoidal $\mathbb{D}$.

Thank you for your attention!



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- *Gambino, Garner, Vasilakopoulou*, “Monoidal Kleisli bicategories and the arithmetic product of symmetric sequences”, Documenta Mathematica (2024)