Stable Independence and Higher Amalgamation

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We consider several recent results connected to the category-theoretic analysis of (nonforking) stable independence, an essential concept in contemporary model theory. In particular:

- Stable independence relations on (nice subcategories of) locally presentable categories correspond precisely to cofibrantly generated WFSs.
- In this rarified context, stable independence in 2 dimensions implies stable independence/amalgamation in all dimensions.
- For categorical model theory, this is too nice a context: even in a stable first order theory, type-amalgamation already fails around dimension 3. Ugly, but interesting: this needs generalizing!

All work is joint with Jiří Rosický and Sebastien Vasey; all baseless speculation is my own.

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$$\mathbf{Mod}(T_{ab}) \hookrightarrow \mathbf{Ab}$$

Here we take all monos, but could take pure, flat, etc., without escaping the realm of model theory.

All kinds of beautiful things happen, of course, but there are costs as well. In particular:

Fact

The category **Ab** *has pushouts;* $\mathbf{Mod}(T_{ab})$ *does not.*

Easier to see: a pushout of monos in **Ab** is not a pushout in $\mathbf{Mod}(T_{ab})$—induced maps will not be mono.

We consider a more general framework, where we choose a family of morphisms $\mathcal{M}$ in a starting category $\mathcal{K}$ that is locally presentable.

Basic problem: Given a locally presentable category $\mathcal{K}$ and family of $\mathcal{K}$-morphisms $\mathcal{M}$, what can we say about

$$\mathcal{K}_{\mathcal{M}}$$

the subcategory of $\mathcal{K}$ whose morphisms are precisely those in $\mathcal{M}$?

Do natural properties of $\mathcal{M}$ correspond to natural properties of $\mathcal{K}_{\mathcal{M}}$?

Note: We assume $\mathcal{M}$ is *normal*—closed under composition, contains all isomorphisms—so $\mathcal{K}_{\mathcal{M}}$ really is a (wide) subcategory of $\mathcal{K}$.

In general, passing to $\mathcal{K}_\mathcal{M}$ expels us from the paradise of locally presentable categories, leaving us with, if we are lucky, accessibility; that is, we are only guaranteed sufficiently directed colimits. Sadly:

Fact

Let $\mathcal{C}$ be accessible with all morphisms mono [and a multi-initial object]. If $\mathcal{C}$ has pushouts, it is small.

So if we engineer $\mathcal{K}_\mathcal{M}$ to be nice, we essentially lose pushouts.

Such is life.

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So, the more we engineer $\mathcal{K}_\mathcal{M}$ to be nice, the less likely we are to have pushouts.

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This is extremely ahistorical...

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Version 1: Fix a theory T, monster model $\mathfrak{C}$. We say the type of a tuple $\bar{a} \in \mathfrak{C}$ over a model B does not fork over $C \subseteq B$ if the type over C has the same complexity, i.e. Morley rank.

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$$\bar{a} \underset{C}{\overset{(\mathfrak{C})}{\downarrow}} B$$

Actual history?

Version 2: Again, given a theory T and monster model $\mathfrak{C}$, we say

$$A \underset{C}{\overset{(\mathfrak{C})}{\perp}} B$$

if the type of any $\bar{a} \in A$ over B does not fork over C. One can think of this as a kind of independence relation: A is independent from B over C.

One can think of $\perp$ as an abstract ternary relation, and axiomatize stable (or *simple*) independence directly.

Actual history?

Version 3: In abstract elementary classes (AECs), we can only work over models, and may not have a monster model. We end up with $\mathrel{\smile\!\!\!\!\mid}$ as a quaternary relation

$$M_1 \overset{M_3}{\underset{M_0}{\mathrel{\smile\!\!\!\!\mid}}} M_2$$

axiomatized as before. In particular, we are picking out a family of diagrams of strong embeddings of the form

$$\begin{array}{ccc} M_1 & \rightarrow & M_3 \\ \uparrow & \mathrel{\smile\!\!\!\!\mid} & \uparrow \\ M_0 & \rightarrow & M_2 \end{array}$$

Idea: Do this in an arbitrary category $\mathcal{K}$.

Definition ([LRV])

An independence notion $\downarrow$ on $\mathcal{K}$ is a family of commutative squares in $\mathcal{K}$ (suitably closed). We say that $\downarrow$ is **weakly stable** *if it satisfies*

1. *Existence: Any span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.*
2. *Uniqueness: there is only one independent square for each span, up to equivalence.*
3. *Transitivity: horizontal and vertical compositions of independent squares are independent.*

Fact

If $\downarrow$ is weakly stable, these squares satisfy the usual cancellation property of pushouts.

To get the analogue of stability, we must impose a locality condition—accessibility now appears.

Consider the category $\mathcal{K}_{\downarrow}$:

- Objects: $f : M \to N$ in $\mathcal{K}$.
- Morphisms: A morphism from $f : M \to N$ to $f' : M' \to N'$ is a $\mathop{\smile}\limits^{|}$-independent square

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \uparrow & \mathop{\smile}\limits^{|} & \uparrow \\ M & \longrightarrow & N \end{array}$$

Definition

*We say that $\mathop{\smile}\limits^{|}$ is λ-**stable** if $\mathcal{K}_{\downarrow}$ is λ-accessible, and **stable** if it is λ-stable for some λ.*

Returning to the basic framework, i.e. $\mathcal{K}$ a category, $\mathcal{M}$ a class of morphisms, there is a natural candidate for stable independence:

Definition

We say a square

$$\begin{array}{ccc} M_1 & \longrightarrow & M_3 \\ \uparrow & & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array}$$

*in $\mathcal{K}$ is $\mathcal{M}$-**effective** if*

1. *all morphisms are in $\mathcal{M}$,*
2. *the pushout of $M_1 \leftarrow M_0 \rightarrow M_2$ exists, and*
3. *the induced map from the pushout to M_3 is in $\mathcal{M}$.*

If $\mathcal{M} = \{\text{regular monos}\}$, these are the *effective unions* of Barr.

To force these squares to form a nice independence relation, we need a few additional properties:

Definition

Let $\mathcal{K}$ be a category.

1. *We say that $\mathcal{M}$ is* **coherent** *if whenever $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, $f \in \mathcal{M}$.*
2. *We say that $\mathcal{M}$ is a* **coclan** *if pushouts of morphisms in $\mathcal{M}$ exist, and $\mathcal{M}$ is closed under pushouts.*
3. *We say $\mathcal{M}$ is* **almost nice** *if it is a coherent coclan, and* **nice** *if, in addition, it is closed under retracts.*

Proposition

If $\mathcal{M}$ is almost nice, the $\mathcal{M}$-effective squares give a weakly stable independence notion on $\mathcal{K}_{\mathcal{M}}$.

Theorem ([LRV2])

Let $\mathcal{K}$ be locally presentable, $\mathcal{M}$ nice and $\aleph_0$-continuous. The following are equivalent:

1. $\mathcal{K}_{\mathcal{M}}$ *has a stable independence notion.*
2. $\mathcal{M}$*-effective squares form a stable independence notion on* $\mathcal{K}_{\mathcal{M}}$*.*
3. $\mathcal{M}$ *is cofibrantly generated.*

Proof.

(1) $\Rightarrow$ (2): By canonicity—clean category-theoretic proof of this. □

Theorem ([LRV2])

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3. *$\mathcal{M}$ is cofibrantly generated.*

Proof.

(2) $\Rightarrow$ (3): Take λ such that $\mathcal{K}_{\mathcal{M},\downarrow}$ and $\mathcal{K}$ are λ-accessible, consider

$$\mathcal{M}_\lambda = \mathcal{M} \cap \mathbf{Pres}_\lambda(\mathbf{C})^{\to}.$$

One can show that $\mathcal{M} = \mathrm{cof}(\mathcal{M}_\lambda)$. □

Theorem ([LRV2])

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Proof.

(3) $\Rightarrow$ (1): Say $\mathcal{M} = \mathrm{cof}(X)$, and λ such that everything is λ-accessible, including domains and codomains of morphisms in X. Show class $\mathcal{M}^*$ of λ-directed colimits of maps in $\mathcal{M}_\lambda$ (in $\mathcal{K}_{\mathcal{M},\downarrow}$) is exactly $\mathcal{M}$. Need elimination of retracts, [MRV]. □

Fact

If $(\mathcal{M},\mathcal{N})$ is a coherent WFS—that is, $\mathcal{M}$ is coherent—then $\mathcal{M}$ is nice and $\aleph_0$-continuous.

Corollary

If $(\mathcal{M},\mathcal{N})$ is a coherent weak factorization system on locally presentable $\mathcal{K}$, the following are equivalent:

1. *$\mathcal{K}_{\mathcal{M}}$ has stable independence.*
2. *$\mathcal{M}$ is cofibrantly generated.*

Note (Quillen's small object argument)

If $\mathcal{K}$ is locally presentable, $\mathcal{M}$ cofibrantly generated, then $(\mathcal{M},\mathcal{M}^{\square})$ is a WFS on $\mathcal{K}$.

So, modulo coherence, subcategories $\mathcal{K}_{\mathcal{M}}$ with stable independence correspond precisely to cofibrantly generated WFSs.

In the coherent case, this gives a quick and easy proof of the pseudopullback theorem of [Makkai/Rosický, '13].

Theorem (Not precise!)

Pseudopullbacks of combinatorial categories $(\mathcal{K}_i, \mathrm{cfib}(\mathcal{K}_i))$, $i = 0, 1, 2$, *in the 2-category of cellular categories are combinatorial.*

Proof.

Pseudopullback in larger 2-category of cellular categories:

$$\begin{array}{ccc} (\mathcal{P}, \mathrm{cfib}(\mathcal{P})) & \longrightarrow & (\mathcal{K}_2, \mathrm{cfib}(\mathcal{K}_2)) \\ \downarrow & & \downarrow \\ (\mathcal{K}_1, \mathrm{cfib}(\mathcal{K}_1)) & \longrightarrow & (\mathcal{K}_0, \mathrm{cfib}(\mathcal{K}_0)) \end{array}$$

□

In the coherent case, this gives a quick and easy proof of the pseudopullback theorem of [Makkai/Rosický, '13].

Theorem (Not precise!)

Pseudopullbacks of combinatorial categories $(\mathcal{K}_i, \mathrm{cfib}(\mathcal{K}_i))$, $i = 0, 1, 2$, *in the 2-category of cellular categories are combinatorial.*

Proof.

Pseudopullback of accessible categories in **CAT**:

$$\begin{array}{ccc} (\mathcal{P})_{\mathrm{cfib}(\mathcal{P})} & \longrightarrow & (\mathcal{K}_2)_{\mathrm{cfib}(\mathcal{K}_2)} \\ \downarrow & & \downarrow \\ (\mathcal{K}_1)_{\mathrm{cfib}(\mathcal{K}_1)} & \longrightarrow & (\mathcal{K}_0)_{\mathrm{cfib}(\mathcal{K}_0)} \end{array}$$

Pseudopullbacks of accessible categories are accessible... □

(Wave hands...)

Definition ([LRV1])

Let $\mathcal{K}$ be a category. For $n \geq 1$, we define an *n-dimensional stable independence relation* on $\mathcal{K}$, Γ, and its induced category $\mathcal{K}_\Gamma$ by induction on n:

- We say Γ is a 1-dimensional stable independence notion on $\mathcal{K}$ just in case it is $\mathrm{Mor}(\mathcal{K})$. In this case, define $\mathcal{K}_\Gamma = \mathcal{K}$.
- An $(n+1)$-dimensional stable independence relation on $\mathcal{K}$ consists of a pair (Γ_n, Γ), where
  1. Γ_n is an n-dimensional stable independence relation on $\mathcal{K}$.
  2. Γ is a stable independence notion on $\mathcal{K}_{\Gamma_n}$
- Given $(n+1)$-dimensional $\Gamma_{n+1} = (\Gamma_n, \Gamma)$ on $\mathcal{K}$, define $\mathcal{K}_{\Gamma_{n+1}} = (\mathcal{K}_{\Gamma_n})_\Gamma$, whose objects are morphisms of $\mathcal{K}_{\Gamma_n}$ and whose morphisms are Γ-independent squares.

That is too much to digest in one sitting, of course. As an exercise, one might check that 2-dimensional stable independence notions correspond to stable independence notions as already defined.

The best case scenario is the following:

Definition

We say that a category $\mathcal{K}$ is *excellent* if for all $n \geq 1$, $\mathcal{K}$ has an n-dimensional stable independence relation Γ_n such that $\mathcal{K}^{\Gamma_n}$ has directed colimits.

We return to our favorite special case: $\mathcal{K}$ locally presentable.

Theorem

Let $\mathcal{K}$ be a locally presentable category, and let $\mathcal{M}$ be a nice, accessible, $\aleph_0$-continuous class of morphisms in $\mathcal{K}$. If $\mathcal{K}_{\mathcal{M}}$ has a stable independence relation, it is excellent.

Proof:

(Sketch) We proceed by induction on dimension.

Recall that, under these hypotheses, $\mathcal{K}_{\mathcal{M}}$ has stable independence just in case $\mathcal{M}$ is cofibrantly generated.

So really, the inductive step involves showing that, given the above assumptions, the class of $\mathcal{M}$-effective morphisms in $\mathcal{K}^2$—call it $\mathcal{M}!$—is well-behaved in exactly the same ways:

- $\mathcal{M}!$ is cofibrantly generated in $\mathcal{K}^2$.
- $\mathcal{M}!$ is nice, $\aleph_0$-continuous, and accessible.

Nearly everything is just bookkeeping, aside from showing $\mathcal{M}!$ is a coclan and that it is cofibrantly generated (easier via stability!).

That means, unfortunately, that this context is far too nice: excellence should not be the default. Pathologies in amalgamation are the norm, rather than the exception.

The issue is that categories arising in model theory are almost never of the form we've been considering. Really:

- We begin with a locally presentable category $\mathbf{Str}(\Sigma)$ of Σ-structures and Σ-homs, Σ a fixed signature, then
- throw away all Σ-structures except those satisfying some theory T in a suitable logic, then
- throw away all but those Σ-monos that preserve a suitable fragment of that logic.

AECs take a syntax-free version of this route, but we still discard lots of Σ-structures.

The second move—when not preceded by the first move!—leads to nice results. When both are at play, things are grittier.

A great deal can be said, though, even if we simply work with (nice...) accessible categories with stable independence:

- Canonicity: if $\mathcal{K}$ is accessible with directed bounds, there is at most one stable independence notion, [LRV]. More...
- Stable 3-amalgamation of models: filling cubes... [KR].
- Induced independence: Under certain conditions functors may preserve or reflect stable independence. In particular, we can say things about when stable independence passes to subcategories, and when it can be pushed upward from a (nice...) subcategory ([LRV1], [KR], ongoing).

There is far more to do to develop the theory of accessible categories with stable independence, and also to explore various notions of *unstable* independence...

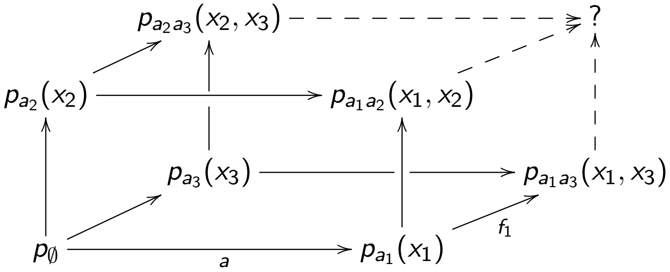
We zoom in on one particular pathology, which requires a shift to classical model theory. Let T be a first order theory. Recall that a *type* $\pi(\bar{x})$ in T is a set of formulas in tuple of variables $\bar{x}$ that is consistent over T, $p_{\bar{a}}(\bar{x})$ the complete type of tuple $\bar{a}$ in T.

Questions

1. *Given types* $\pi_1(x_1), \pi_2(x_2)$*, is there a (unique) minimal consistent extension?*
2. *What about* $\pi_{12}(x_1, x_2), \pi_{13}(x_1, x_3), \pi_{23}(x_2, x_3)$*?*
3. *...*
4. *What about* $\pi_{23\ldots n}(x_2, x_3, \ldots, x_n), \pi_{13\ldots n}(x_1, x_3, \ldots, x_n), \ldots, \pi_{12\ldots(n-1)}(x_1, x_2, \ldots, x_{n-1})$*?*

The answer to every single one of these, as currently phrased, is no. To rule out stupid obstacles, we restrict the problem significantly.

Let $(a_1, \ldots, a_n)$ be an independent n-tuple. We build an incomplete n-cube of complete types of subtuples, beginning with $p_\emptyset$, complete types of sub-1-tuples, sub-2-tuples, etc. For $n = 3$:



Can this be filled so nothing forks anywhere (*n-amalgamation*)? In exactly one way (*n-uniqueness*)?

Theorem (Shelah)

If T is stable, T has 2-uniqueness.

"Recall" that given T, T^{eq} is an extension of T that permits the elimination of imaginaries. (Folklore: This corresponds to taking the pretopos completion of the syntactic category. Almost true.)

Theorem (Hrushovski, [H])

If T is stable and $T = T^{eq}$, TFAE:

1. *T has 4-amalgamation.*
2. *T has 3-uniqueness.*
3. *Any T-definable connected groupoid is eliminable: T thinks it's actually a group.*

Corollary

Failures of 4-amalgamation are witnessed by (definable) pathological groupoids.

This work inspired a series of papers by model theorists Goodrick, Kim and Kolesnikov linking failure in dimension n to the existence of definable, non-eliminable higher groupoids: specifically, what they call n-ary poly- (or quasi-) groupoids.

They note that they are filling cubes, and that this probably has something to do with $(\infty, 1)$-categories. No one with the capacity to pursue this observation has ever done so... or, I suspect, even encountered it.

Since we can give a clean account of stable independence on an abstract category, can we not analyse this problem with less fuss, and no (classical) syntax?

There are clear paths to notions of type meaningful in this context:

- Passing to the topos of types, e.g. via Makkai's construction.
- Considering (abstract) Galois types. . . could work.

To what extent do the phenomena we've just described appear? How does elimination of imaginaries *really* manifest itself?

This is the subject of ongoing—and very preliminary—work.

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