Lifting independence

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That is, a well-defined cardinal $\dim(V)$ can be assigned to each $\mathbb{Q}$-vector space V and there is exactly one $\mathbb{Q}$-vector space with dimension κ for each cardinal κ.

Stability theory

$\mathbb{Q}$-vector spaces and algebraically closed fields

Can we characterise which $\mathbb{Q}$-vector spaces there are (up to isomorphism)?

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That is, a well-defined cardinal $\dim(V)$ can be assigned to each $\mathbb{Q}$-vector space V and there is exactly one $\mathbb{Q}$-vector space with dimension κ for each cardinal κ.

Similarly, algebraically closed fields (of a fixed characteristic) are determined by their transcendence degree.

Stability theory

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This applies to our examples of $\mathbb{Q}$-vector spaces and algebraically closed fields.

This sparked a great amount of research with as a result Shelah's celebrated *stability theory* (1970).

Stability theory

Main gap theorem

Write $I(T, \kappa)$ for the number of models of a theory T of cardinality κ.

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So Morley's theorem says that if $I(T,\kappa)=1$ for some uncountable κ then $I(T,\kappa)=1$ for all uncountable κ (T countable).

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So Morley's theorem says that if $I(T, \kappa) = 1$ for some uncountable κ then $I(T, \kappa) = 1$ for all uncountable κ (T countable).

Theorem (Main Gap Theorem, Shelah (1982))

Let T be a countable theory. Either $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ for all $\alpha \geq 1$ (i.e. it is maximal) or

$$I(T, \aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$$

for all $\alpha \geq 1$.

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Side note: it is actually the negation, so non-forking, that gives us independence. So forking actually expresses that things are dependent.

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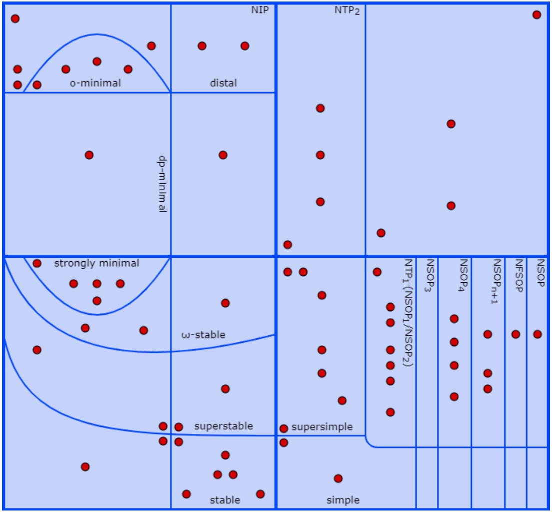
Side note: it is actually the negation, so non-forking, that gives us independence. So forking actually expresses that things are dependent.

Linear independence in vector spaces coincides with non-forking. Same for algebraic independence in algebraically closed fields.

Shelah pinned down a class of theories where forking is very well-behaved, the *stable* theories.

Stability theory

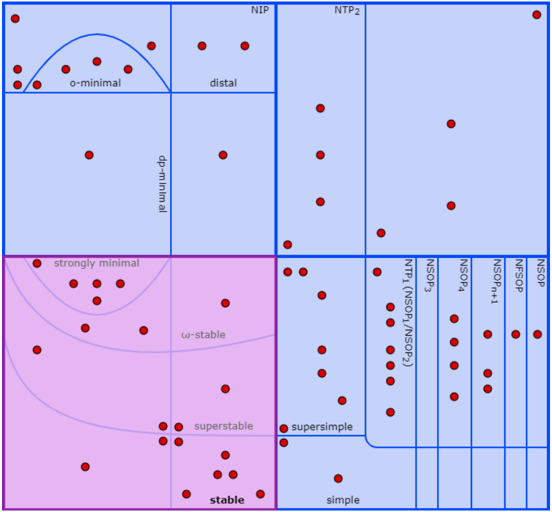
A map of the universe



Source: `https://forkinganddividing.com`

Stability theory

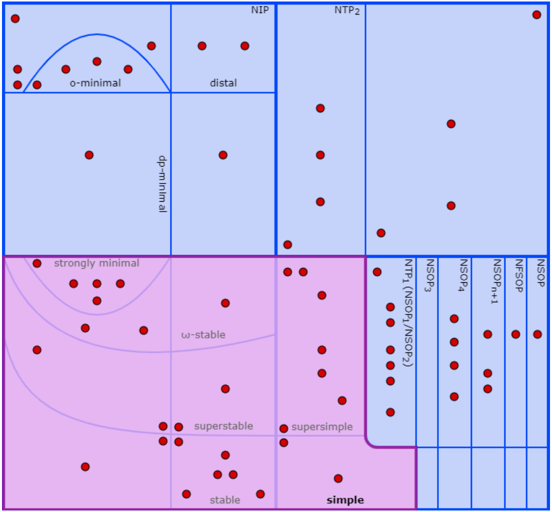
A map of the universe - Stable



Source: `https://forkinganddividing.com`

Stability theory

A map of the universe - Simple



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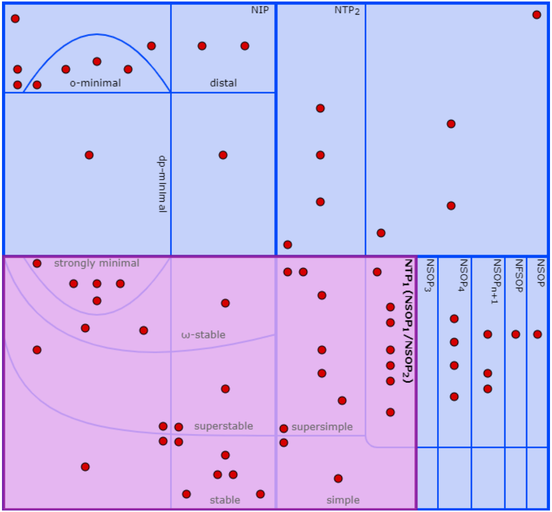
Stability theory

A map of the universe - $NSOP_1$



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Stability theory

Simple and $NSOP_1$

Kim and Pillay generalised the work on forking to the class of *simple* theories, where it is still reasonably well-behaved (1997).

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In $NSOP_1$ theories forking is no longer so well-behaved, yet many $NSOP_1$ theories with a good notion of independence were known.

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In $NSOP_1$ theories forking is no longer so well-behaved, yet many $NSOP_1$ theories with a good notion of independence were known.

Kaplan and Ramsey, inspired by ideas from Kim, developed a notion called *Kim-forking*, which is well-behaved in $NSOP_1$ theories (2017).

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Kim and Pillay generalised the work on forking to the class of *simple* theories, where it is still reasonably well-behaved (1997).

In $NSOP_1$ theories forking is no longer so well-behaved, yet many $NSOP_1$ theories with a good notion of independence were known.

Kaplan and Ramsey, inspired by ideas from Kim, developed a notion called *Kim-forking*, which is well-behaved in $NSOP_1$ theories (2017).

The ‘good’ case of Shelah’s main gap theorem takes place in the stable class (even superstable). However, the tools developed for it are still useful in the simple and $NSOP_1$ classes. So it is interesting to know where a theory lives in this picture.

Stability theory

Classification based on independence

We can define an independence relation to be *stable* if it satisfies a certain list of properties.

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Similarly, an independence relation is *simple* (resp. *$NSOP_1$-like*) if it satisfies the same list of properties, minus one (resp. minus two) specific properties.

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Classification based on independence

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Similarly, an independence relation is *simple* (resp. *$NSOP_1$-like*) if it satisfies the same list of properties, minus one (resp. minus two) specific properties.

Theorem

A first-order theory T is stable/simple/$NSOP_1$ iff there is a stable/simple/$NSOP_1$-like independence relation. Furthermore, this independence relation is unique.

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Independence relations

Let V be an $\mathbb{Q}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{V} B \iff \operatorname{span}(A \cup C) \cap \operatorname{span}(B \cup C) \subseteq \operatorname{span}(C).$$

We say that A *is independent from* B *over* C.

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We have that $a_1, \ldots, a_n \in V$ are linearly independent iff $a_i \downarrow^V_\emptyset a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ for all $1 \leq i \leq n$.

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We call $\downarrow$ an *independence relation*.

Independence relations

Properties

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For $\mathbb{Q}$-vector spaces $\mathop{\smile\!\!\!\!\!\mid}$ has nice properties.

If $A \mathop{\smile\!\!\!\!\!\mid}_C^V B$ then also $B \mathop{\smile\!\!\!\!\!\mid}_C^V A$ (symmetry).

If $B' \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\mid}_C^V B$ then also $A \mathop{\smile\!\!\!\!\!\mid}_C^V B'$ (monotonicity).

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And some more...

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Setup

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Categorical approach

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A category satisfying the first three items is called an *AECat*, which is short for *Abstract Elementary Category* (Kamsma [2020]).

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The final item is considered a separate property, and is abbreviated to *AP*.

Categorical approach

Examples

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1. For a first-order theory T the category $\mathbf{Mod}(T)$ is an AECat with AP.
2. For a positive theory T we let $\mathbf{Mod}(T)$ be the category of p.c. models and embeddings, then again $\mathbf{Mod}(T)$ is an AECat with AP.

Categorical approach

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3. For any continuous theory T we can form $\mathbf{MetMod}(T)$, which is an AECat with AP.

Categorical approach

Examples

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2. For a positive theory T we let $\mathbf{Mod}(T)$ be the category of p.c. models and embeddings, then again $\mathbf{Mod}(T)$ is an AECat with AP.
3. For any continuous theory T we can form $\mathbf{MetMod}(T)$, which is an AECat with AP.
4. Any Abstract Elementary Class (AEC) $\mathcal{K}$ can be viewed as a category by taking as arrows $\mathcal{K}$-embeddings: that is, embeddings $f : M \to N$ such that $f(M) \leq_{\mathcal{K}} N$. Then $\mathcal{K}$ is an AECat and the definition of AP coincides with the definition of "amalgamation property" as it is usually stated for AECs.

Categorical approach

Categorical independence

Using the ideas from Lieberman et al. [2019].

Definition

An *independence relation* $\downarrow$ on a category $\mathcal{C}$ is a relation on commuting squares in $\mathcal{C}$. If a square is in the relation we call it *independent* and write

$$\begin{array}{ccc} A & \longrightarrow & D \\ \uparrow & \downarrow & \uparrow \\ C & \longrightarrow & B \end{array}$$

Categorical approach

Categorical independence

Using the ideas from Lieberman et al. [2019].

Definition

An *independence relation* $\downfree$ on a category $\mathcal{C}$ is a relation on commuting squares in $\mathcal{C}$. If a square is in the relation we call it *independent* and write

$$\begin{array}{ccc} A & \longrightarrow & D \\ \uparrow & \downfree & \uparrow \\ C & \longrightarrow & B \end{array}$$

Throughout, we should think of $\mathcal{C}$ as an AECat.

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Concrete example

Fix a field K and consider $\mathbf{Bil}_K$, the category of bilinear spaces over K with injective bilinear morphisms (i.e., injective linear maps that respect the bilinear form).

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Fix a field K and consider $\mathbf{Bil}_K$, the category of bilinear spaces over K with injective bilinear morphisms (i.e., injective linear maps that respect the bilinear form).

Let $\mathbf{Vec}_K$ be the category of vector spaces over K with injective linear maps and recall that we had an independence relation $\perp$ on $\mathbf{Vec}_K$ given by linear independence.

Lifting independence

Concrete example

Fix a field K and consider $\mathbf{Bil}_K$, the category of bilinear spaces over K with injective bilinear morphisms (i.e., injective linear maps that respect the bilinear form).

Let $\mathbf{Vec}_K$ be the category of vector spaces over K with injective linear maps and recall that we had an independence relation $\downfree$ on $\mathbf{Vec}_K$ given by linear independence.

The canonical independence relation on $\mathbf{Bil}_K$ turns out to be same. That is, given by linear independence.

Lifting independence

Concrete example - categorical perspective

Write $F : \mathbf{Bil}_K \to \mathbf{Vec}_K$ for the forgetful functor.

Lifting independence

Concrete example - categorical perspective

Write $F : \mathbf{Bil}_K \to \mathbf{Vec}_K$ for the forgetful functor.

Rephrasing the previous slide: the square below on the left (in $\mathbf{Bil}_K$) is independent iff the square below on the right (in $\mathbf{Vec}_K$) is independent.

$$\begin{array}{ccc} A & \longrightarrow & D \\ \uparrow & & \uparrow \\ C & \longrightarrow & B \end{array} \qquad\qquad \begin{array}{ccc} F(A) & \longrightarrow & F(D) \\ \uparrow & & \uparrow \\ F(C) & \longrightarrow & F(B) \end{array}$$

Lifting independence

Basics

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\perp$ be an independence relation on $\mathcal{D}$. We define the *lift* $F^{-1}(\perp)$ of $\perp$ along F as follows. A commuting square in $\mathcal{C}$ is $F^{-1}(\perp)$-independent if and only if its image under F is $\perp$-independent.

Lifting independence

Basics

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation on $\mathcal{D}$. We define the *lift* $F^{-1}(\mathop{\smile\!\!\!\!\!\!|})$ of $\mathop{\smile\!\!\!\!\!\!|}$ along F as follows. A commuting square in $\mathcal{C}$ is $F^{-1}(\mathop{\smile\!\!\!\!\!\!|})$-independent if and only if its image under F is $\mathop{\smile\!\!\!\!\!\!|}$-independent.

Question: what properties of $\mathop{\smile\!\!\!\!\!\!|}$ does $F^{-1}(\mathop{\smile\!\!\!\!\!\!|})$ inherit? What (reasonable) assumptions can we place on F to add more properties to this list?

Lifting independence

Basics

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation on $\mathcal{D}$. We define the *lift* $F^{-1}(\mathop{\smile\!\!\!\!\!\!|})$ of $\mathop{\smile\!\!\!\!\!\!|}$ along F as follows. A commuting square in $\mathcal{C}$ is $F^{-1}(\mathop{\smile\!\!\!\!\!\!|})$-independent if and only if its image under F is $\mathop{\smile\!\!\!\!\!\!|}$-independent.

Question: what properties of $\mathop{\smile\!\!\!\!\!\!|}$ does $F^{-1}(\mathop{\smile\!\!\!\!\!\!|})$ inherit? What (reasonable) assumptions can we place on F to add more properties to this list?

Proposition

Any functor will lift the properties invariance, monotonicity, symmetry, transitivity and basic existence.

Lifting independence

Accessibility and union

Definition

Let $\mathop{\smile}\limits^{|}$ be an independence relation on a category $\mathcal{C}$. If $\mathop{\smile}\limits^{|}$ satisfies transitivity and basic existence then we can form the subcategory $\mathcal{C}_{\mathop{\smile}\limits^{|}}$ of $\mathcal{C}^2$ with the same objects, but whose morphisms are restricted to $\mathop{\smile}\limits^{|}$-independent squares. We then say that:

Lifting independence

Accessibility and union

Definition

Let $\downarrow$ be an independence relation on a category $\mathcal{C}$. If $\downarrow$ satisfies transitivity and basic existence then we can form the subcategory $\mathcal{C}_{\downarrow}$ of $\mathcal{C}^2$ with the same objects, but whose morphisms are restricted to $\downarrow$-independent squares. We then say that:

- $\downarrow$ is *accessible* if $\mathcal{C}_{\downarrow}$ is an accessible category,

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Accessibility and union

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Let $\downarrow$ be an independence relation on a category $\mathcal{C}$. If $\downarrow$ satisfies transitivity and basic existence then we can form the subcategory $\mathcal{C}_{\downarrow}$ of $\mathcal{C}^2$ with the same objects, but whose morphisms are restricted to $\downarrow$-independent squares. We then say that:

- $\downarrow$ is *accessible* if $\mathcal{C}_{\downarrow}$ is an accessible category,
- $\downarrow$ satisfies *union* if $\mathcal{C}_{\downarrow}$ has directed colimits and these are preserved by the inclusion functor $\mathcal{C}_{\downarrow} \hookrightarrow \mathcal{C}^2$.

Lifting independence

Accessibility and union

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a directed colimit preserving functor between accessible categories with directed colimits. Suppose that $\downarrow$ is an independence relation on $\mathcal{D}$ satisfying transitivity and basic existence.

Lifting independence

Accessibility and union

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a directed colimit preserving functor between accessible categories with directed colimits. Suppose that $\perp$ is an independence relation on $\mathcal{D}$ satisfying transitivity and basic existence.

1. *If $\perp$ satisfies union then so does $F^{-1}(\perp)$.*

Lifting independence

Accessibility and union

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a directed colimit preserving functor between accessible categories with directed colimits. Suppose that $\downarrow$ is an independence relation on $\mathcal{D}$ satisfying transitivity and basic existence.

1. *If $\downarrow$ satisfies union then so does $F^{-1}(\downarrow)$.*
2. *If $\downarrow$ satisfies union and is accessible then the same holds for $F^{-1}(\downarrow)$.*

Lifting independence

All properties

	Stable	Simple	$NSOP_1$-like
Invariance	✓	✓	✓
Monotonicity	✓	✓	✓
Symmetry	✓	✓	✓
Transitivity	✓	✓	✓
Basic existence	✓	✓	✓
Union	✓	✓	✓
Accessible	✓	✓	✓
Existence	✓	✓	✓
3-amalgamation	✓	✓	✓
Base monotonicity	✓	✓	
Uniqueness	✓		

Lifting independence

Lifting uniqueness (attempt)

Proposition

If $F : \mathcal{C} \to \mathcal{D}$ is a left multiadjoint and $\downarrow$ is an independence relation on $\mathcal{D}$ that satisfies uniqueness then $F^{-1}(\downarrow)$ satisfies uniqueness.

Lifting independence

Lifting uniqueness (attempt)

Proposition

If $F : \mathcal{C} \to \mathcal{D}$ is a left multiadjoint and $\downarrow$ is an independence relation on $\mathcal{D}$ that satisfies uniqueness then $F^{-1}(\downarrow)$ satisfies uniqueness.

Left (multi)adjoints that look like forgetful functors are rare, definitely between AECats.

Lifting independence

Left (multi)adjoints between bigger categories

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Meanwhile, $\mathbf{Ab}_{\text{mono}}$ carries a stable independence relation given by pullback squares.

Lifting independence

Left (multi)adjoints between bigger categories

Often left (multi)adjoints are functors between bigger categories.

For example the forgetful $\mathbf{Ab}^M \to \mathbf{Ab}$, where M is a monoid, is left adjoint.

Meanwhile, $\mathbf{Ab}_{\text{mono}}$ carries a stable independence relation given by pullback squares.

The forgetful $\mathbf{Ab}^M \to \mathbf{Ab}$ restricts to $\mathbf{Ab}^M_{\text{mono}} \to \mathbf{Ab}_{\text{mono}}$, and the latter lifts uniqueness, as is seen by temporarily working in the bigger categories.

Lifting independence

Lifting uniqueness

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a left multiadjoint and let $\mathcal{M}$ be a left-cancellable composable class of arrows in $\mathcal{D}$. If $\downarrow$ is an independence relation on $\mathcal{D}_{\mathcal{M}}$ that satisfies uniqueness then the independence relation $F^{-1}(\downarrow)$ on $\mathcal{C}_{F^{-1}(\mathcal{M})}$ satisfies uniqueness.

Lifting independence

Lifting everything

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful left multiadjoint and let $\mathcal{M}$ be a left-cancellable composable accessible and continuous class of monomorphisms in $\mathcal{D}$. Suppose that $\downarrow$ is an independence relation on $\mathcal{D}_{\mathcal{M}}$, that satisfies semi-invariance as an independence relation on $\mathcal{D}$.

1. *If $\downarrow$ is stable then $F^{-1}(\downarrow)$ is stable.*
2. *If $\downarrow$ is simple, $\mathcal{C}_{F^{-1}(\mathcal{M})}$ and $\mathcal{D}_{\mathcal{M}}$ have binary joins of subobjects and F preserves those then then $F^{-1}(\downarrow)$ is simple.*
3. *If $\downarrow$ is NSOP$_1$-like then $F^{-1}(\downarrow)$ is NSOP$_1$-like.*

Lifting independence

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Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful left multiadjoint and let $\mathcal{M}$ be a left-cancellable composable accessible and continuous class of monomorphisms in $\mathcal{D}$. Suppose that $\mathop{\smile\!\!\!\!\!\!|}\,$ is an independence relation on $\mathcal{D}_{\mathcal{M}}$, that satisfies semi-invariance as an independence relation on $\mathcal{D}$.

1. *If $\mathop{\smile\!\!\!\!\!\!|}\,$ is stable then $F^{-1}(\mathop{\smile\!\!\!\!\!\!|}\,)$ is stable.*
2. *If $\mathop{\smile\!\!\!\!\!\!|}\,$ is simple, $\mathcal{C}_{F^{-1}(\mathcal{M})}$ and $\mathcal{D}_{\mathcal{M}}$ have binary joins of subobjects and F preserves those then then $F^{-1}(\mathop{\smile\!\!\!\!\!\!|}\,)$ is simple.*
3. *If $\mathop{\smile\!\!\!\!\!\!|}\,$ is NSOP$_1$-like then $F^{-1}(\mathop{\smile\!\!\!\!\!\!|}\,)$ is NSOP$_1$-like.*

There is a second flavour of conditions on F in our preprint (Kamsma and Rosický [2024]) that gives a similar conclusion.

References

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