Simplicial completion of (pre)model categories and strictification (Joint work in progress with Chaitanya Leena Subramaniam)

Simon Henry

University of Ottawa

Decembre 18th, 2024

Quillen model categories are a very convenient way to organise a “Homotopy theories”.

Quillen model categories are a very convenient way to organise a “Homotopy theories”.

But simplicial model categories are often more convenient.

Quillen model categories are a very convenient way to organise a "Homotopy theories".

But simplicial model categories are often more convenient. For example, If $\mathcal{C}$ is a simplicial model category, then the simplicial category of bifibrant object is directly a model for the Dwyer-Kan localization of $\mathcal{C}$, i.e. the ∞-category modeled by $\mathcal{C}$.

Quillen model categories are a very convenient way to organise a "Homotopy theories".

But simplicial model categories are often more convenient. For example, If $\mathcal{C}$ is a simplicial model category, then the simplicial category of bifibrant object is directly a model for the Dwyer-Kan localization of $\mathcal{C}$, i.e. the ∞-category modeled by $\mathcal{C}$.

Simplicial completion is a process to turn a general model category $\mathcal{C}$ into a simplicial one.

Quillen model categories are a very convenient way to organise a "Homotopy theories".

But simplicial model categories are often more convenient. For example, If $\mathcal{C}$ is a simplicial model category, then the simplicial category of bifibrant object is directly a model for the Dwyer-Kan localization of $\mathcal{C}$, i.e. the ∞-category modeled by $\mathcal{C}$.

Simplicial completion is a process to turn a general model category $\mathcal{C}$ into a simplicial one. This is done by considering the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$.

Theorem (Dugger)

If C is a left proper combinatorial Quillen model category, then the category sC of simplicial objects of C has a simplicial model structure such that:

- *The constant object functor $C \to sC$ is a left Quillen equivalence.*
- *Any left Quillen functor $C \to D$ with D a simplicial model category can be factored uniquely into a simplicial left Quillen functor $C \to sC \to D$.*

Let's quickly explain how this works:

Let's quickly explain how this works:
First, sC is a simplicially enriched, tensored and cotensored category:

$$(A \otimes X)_k = A_k \times X_k = \coprod_{A_k} X_k \quad (A \in sSet, \text{ and } X \in sC)$$

Let's quickly explain how this works:
First, sC is a simplicially enriched, tensored and cotensored category:

$$(A \otimes X)_k = A_k \times X_k = \coprod_{A_k} X_k \quad (A \in sSet, \text{ and } X \in sC)$$

$$\mathrm{Hom}^{\widehat{\Delta}}(X, Y)_k = \int_{[n] \in \Delta} \mathrm{Hom}\,(X([n]), Y([n]))^{\Delta([n],[k])} \qquad Y^A = \int_{[n] \in \Delta} Y([n])^{A([n])}$$

Let's quickly explain how this works:
First, sC is a simplicially enriched, tensored and cotensored category:

$$(A \otimes X)_k = A_k \times X_k = \coprod_{A_k} X_k \quad (A \in sSet, \text{ and } X \in sC)$$

$$\mathrm{Hom}^{\widehat{\Delta}}(X, Y)_k = \int_{[n] \in \Delta} \mathrm{Hom}\,(X([n]), Y([n]))^{\Delta([n],[k])} \qquad Y^A = \int_{[n] \in \Delta} Y([n])^{A([n])}$$

We will identify C with the subcategory of sC of constant objects.

The class of cofibrations and trivial cofibrations of sC are generated by the cofibrations and trivial cofibrations of C, The class of cofibrations and trivial cofibrations of sC are generated by the cofibrations and trivial cofibrations of C, but as a simplicial model category,

The class of cofibrations and trivial cofibrations of sC are generated by the cofibrations and trivial cofibrations of C, but as a simplicial model category, that is we need the following maps as generators:

- Generating cofibrations:

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

For $A \to B$ any (generating) cofibration of C.

The class of cofibrations and trivial cofibrations of sC are generated by the cofibrations and trivial cofibrations of C, but as a simplicial model category, that is we need the following maps as generators:

- Generating cofibrations:

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

  For $A \to B$ any (generating) cofibration of C.

- (Pseudo) Generating trivial cofibrations:

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

  for $A \to B$ a (generating) trivial cofibration of C

The class of cofibrations and trivial cofibrations of sC are generated by the cofibrations and trivial cofibrations of C, but as a simplicial model category, that is we need the following maps as generators:

- Generating cofibrations:

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

  For $A \to B$ any (generating) cofibration of C.
- (Pseudo) Generating trivial cofibrations:

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

  for $A \to B$ a (generating) trivial cofibration of C and

$$\Lambda^k[n] \otimes B \coprod_{\Lambda^k[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

  For $A \to B$ a (generating) cofibration of C.

Key point of the proof of Dugger's theorem:

Key point of the proof of Dugger's theorem:
If we drop the second familly of of the trivial cofibrations, and only keep the generators of the form

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

Key point of the proof of Dugger's theorem:
If we drop the second familly of of the trivial cofibrations, and only keep the generators of the form

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

What we get is exactly the Reedy model structure on $sC = C^{\Delta^{op}}$.

Key point of the proof of Dugger's theorem:
If we drop the second familly of of the trivial cofibrations, and only keep the generators of the form

$$\partial\Delta[n] \otimes B \coprod_{\partial\Delta[n]\otimes A} \Delta[n] \otimes A \to \Delta[n] \otimes B$$

What we get is exactly the Reedy model structure on $sC = C^{\Delta^{op}}$.
The model structure on sC is constructed as a left Bousfield localization of the Reedy model structure whose local objects are the Reedy fibrant homotopically constant diagram $\Delta^{op} \to C$.
It is easy to see that this is Quillen equivalent to C.

In the previous arguement, the only reason we need C to be left proper, is in order to use the theory of Bousfield localization.

In the previous arguement, the only reason we need C to be left proper, is in order to use the theory of Bousfield localization. But Barwick and Batanin - White have shown that if we use "left semi-model categories", then the theory of Bousfield localization works without the properness assumption.

In the previous arguement, the only reason we need C to be left proper, is in order to use the theory of Bousfield localization. But Barwick and Batanin - White have shown that if we use "left semi-model categories", then the theory of Bousfield localization works without the properness assumption. So we get:

Theorem (Dugger + Barwick, Batanin - White)

If C is a combinatorial Quillen model category, then the category sC of simplicial objects of C has a simplicial left semi-model structure such that:

- *The constant object functor $C \to sC$ is a left Quillen equivalence.*
- *Any left Quillen functor $C \to D$ with D a simplicial left semi-model category can be factored uniquely into a simplicial left Quillen functor $C \to sC \to D$.*

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Combinatorial premodel categories and Quillen functors between premodel cateogires are defined in the same way as as for Quillen model categories.

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Combinatorial premodel categories and Quillen functors between premodel cateogires are defined in the same way as as for Quillen model categories.

Premodel category are in general not homotopy theoretic objects.

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Combinatorial premodel categories and Quillen functors between premodel cateogires are defined in the same way as as for Quillen model categories.

Premodel category are in general not homotopy theoretic objects. But maybe surprisingly, their simplicial completion is:

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Combinatorial premodel categories and Quillen functors between premodel cateogires are defined in the same way as as for Quillen model categories.

Premodel category are in general not homotopy theoretic objects. But maybe surprisingly, their simplicial completion is:

Theorem (Folk?)

If C is a combinatorial premodel category, then the category sC of simplicial objects of C has a simplicial left semi-model structure such that:

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Combinatorial premodel categories and Quillen functors between premodel cateogires are defined in the same way as as for Quillen model categories.

Premodel category are in general not homotopy theoretic objects. But maybe surprisingly, their simplicial completion is:

Theorem (Folk?)

If C is a combinatorial premodel category, then the category sC of simplicial objects of C has a simplicial left semi-model structure such that:

- *The constant object functor $C \to sC$ is a left Quillen ~~equivalence~~ functor.*

Definition (Barton)

A Premodel category is a complete and cocomplete category with two weak factorization systems (cofibrations, anodyne fibrations) and (anodyne cofibrations, fibrations), such that anodyne (co)fibrations are (co)fibrations.

Combinatorial premodel categories and Quillen functors between premodel cateogires are defined in the same way as as for Quillen model categories.

Premodel category are in general not homotopy theoretic objects. But maybe surprisingly, their simplicial completion is:

Theorem (Folk?)

If C is a combinatorial premodel category, then the category sC of simplicial objects of C has a simplicial left semi-model structure such that:

- *The constant object functor $C \to sC$ is a left Quillen ~~equivalence~~ functor.*
- *Any left Quillen functor $C \to D$ with D a simplicial left semi-model category can be factored uniquely into a simplicial left Quillen functor $C \to sC \to D$.*

This leads to the main question we wanted to answer:

This leads to the main question we wanted to answer: If C is a model category, then we know what sC models: it is Quillen equivalent to C. But for a premodel C, then sC is a brand new model category. We want to understand what it models.

This leads to the main question we wanted to answer: If C is a model category, then we know what sC models: it is Quillen equivalent to C. But for a premodel C, then sC is a brand new model category. We want to understand what it models.

Theorem (H. - L. S.)

Let C be a κ-combinatorial premodel category. Let $Cof_\kappa\, C$ be the category of κ-presentable cofibrant objects.

This leads to the main question we wanted to answer: If C is a model category, then we know what sC models: it is Quillen equivalent to C. But for a premodel C, then sC is a brand new model category. We want to understand what it models.

Theorem (H. - L. S.)

Let C be a κ-combinatorial premodel category. Let $\mathrm{Cof}_\kappa\, C$ be the category of κ-presentable cofibrant objects. Then the ∞-category associated to sC is equivalent to the ∞-category of functors $F : (\mathrm{Cof}_\kappa\, C)^{op} \to \mathrm{Space}$ such that

This leads to the main question we wanted to answer: If C is a model category, then we know what sC models: it is Quillen equivalent to C. But for a premodel C, then sC is a brand new model category. We want to understand what it models.

Theorem (H. - L. S.)

Let C be a κ-combinatorial premodel category. Let $Cof_\kappa\, C$ be the category of κ-presentable cofibrant objects. Then the ∞-category associated to sC is equivalent to the ∞-category of functors $F : (Cof_\kappa\, C)^{op} \to Space$ such that

1. *F sends the initial object to $\{*\}$.*
2. *F sends pushout along cofibrations to pullback.*
3. *F sends κ-small transfinite composition of cofibration to limits.*
4. *F sends anodyne cofibrations to weak equivalences.*

A "generalized algebraic theory" in the sense of Cartmell is a kind of multi-sorted algebraic theory where the type/sort can depends on variable taken from other sorts.

Example

The Generalized algebraic theory (GAT) of categories:

A "generalized algebraic theory" in the sense of Cartmell is a kind of multi-sorted algebraic theory where the type/sort can depends on variable taken from other sorts.

Example

The Generalized algebraic theory (GAT) of categories: Type axioms:

$$\vdash \mathrm{Ob}\ \mathrm{Type} \qquad x, y : \mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \mathrm{Type}$$

A "generalized algebraic theory" in the sense of Cartmell is a kind of multi-sorted algebraic theory where the type/sort can depends on variable taken from other sorts.

Example

The Generalized algebraic theory (GAT) of categories: Type axioms:

$$\vdash \mathrm{Ob}\ \mathrm{Type} \qquad x, y : \mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \mathrm{Type}$$

Term axioms:

$$x : \mathrm{Ob} \vdash \mathrm{Id}_x : \mathrm{Hom}(x, x)$$

$$x, y, z : \mathrm{Ob}; f : \mathrm{Hom}(x, y); g : \mathrm{Hom}(y, z) \vdash g \circ_{x,y,z} f : \mathrm{Hom}(x, z))$$

A "generalized algebraic theory" in the sense of Cartmell is a kind of multi-sorted algebraic theory where the type/sort can depends on variable taken from other sorts.

Example

The Generalized algebraic theory (GAT) of categories: Type axioms:

$$\vdash \mathrm{Ob}\ \mathrm{Type} \qquad x, y : \mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \mathrm{Type}$$

Term axioms:

$$x : \mathrm{Ob} \vdash \mathrm{Id}_x : \mathrm{Hom}(x, x)$$

$$x, y, z : \mathrm{Ob}; f : \mathrm{Hom}(x, y); g : \mathrm{Hom}(y, z) \vdash g \circ_{x,y,z} f : \mathrm{Hom}(x, z))$$

Term Equality axioms:

$$x, y : \mathrm{Ob}; f : \mathrm{Hom}(x, y) \vdash f \circ_{x,x,y} \mathrm{Id}_x = f \qquad x, y : \mathrm{Ob}; f : \mathrm{Hom}(x, y) \vdash \mathrm{Id}_y \circ_{x,y,y} f = f$$

$$w, x, y, z : \mathrm{Ob}; f : \mathrm{Hom}(w, x); g : \mathrm{Hom}(x, y); k : \mathrm{Hom}(y, z) \vdash k \circ (g \circ f) = (k \circ g) \circ f$$

Given such a theory T, we can form a syntactic category C_T of T, whose objects are "context" and whose morphism are definable "function" between such context up to provable equality

Given such a theory T, we can form a syntactic category C_T of T, whose objects are "context" and whose morphism are definable "function" between such context up to provable equality

Example

$$\Delta = (a, b : \mathrm{Ob}; v : \mathrm{Hom}(a, b))$$

$$\Gamma = (x, y, z : \mathrm{Ob}; g, f : \mathrm{Hom}(x, y); h : \mathrm{Hom}(y, x); k : \mathrm{Hom}(z, x))$$

are contexts.

$$\begin{array}{ccc} \Gamma & \to & \Delta \\ (x, y, z, g, f, h, k) & \mapsto & (z, y, f \circ k) \end{array}$$

is a context morphism.

This category C_T has a special class of maps that I will call "display maps" that corresponds to just forgetting the last variable from the context. For example:

$$\begin{array}{ccc} \Delta & \to & (a, b : \mathrm{Ob}) \\ (a, b, v) & \mapsto & (a, b) \end{array}$$

This category C_T has a special class of maps that I will call “display maps” that corresponds to just forgetting the last variable from the context. For example:

$$\begin{array}{ccc} \Delta & \rightarrow & (a, b : \mathrm{Ob}) \\ (a, b, v) & \mapsto & (a, b) \end{array}$$

These maps are closed under pullbacks.

This category C_T has a special class of maps that I will call "display maps" that corresponds to just forgetting the last variable from the context. For example:

$$\begin{array}{ccc} \Delta & \to & (a, b : \mathrm{Ob}) \\ (a, b, v) & \mapsto & (a, b) \end{array}$$

These maps are closed under pullbacks.

Theorem (Cartmell)

The category of models of T is equivalent to the category of functor $C_T \to Set$ preserving the terminal object and pullback along display maps.

In particular, the Yoneda embedding define a fully faithful functor $Y : C_T^{op} \to \mathrm{Mod}(T) \subset \mathrm{Fun}(C_T, \mathrm{Set})$.

In particular, the Yoneda embedding define a fully faithful functor $Y : C_T^{op} \to \text{Mod}(T) \subset \text{Fun}(C_T, \text{Set})$.

We consider the weak factorization on $\text{Mod}(T)$ cofibrantly generated by the map $Y(A) \to Y(B)$ for $B \twoheadrightarrow A$ a display map.

In particular, the Yoneda embedding define a fully faithful functor $Y : C_T^{op} \to \mathrm{Mod}(T) \subset \mathrm{Fun}(C_T, \mathrm{Set})$.

We consider the weak factorization on $\mathrm{Mod}(T)$ cofibrantly generated by the map $Y(A) \to Y(B)$ for $B \twoheadrightarrow A$ a display map.

We call the left class “cofibrations” and the right class “Anodyne fibrations”.

In particular, the Yoneda embedding define a fully faithful functor $Y : C_T^{op} \to \mathrm{Mod}(T) \subset \mathrm{Fun}(C_T, \mathrm{Set})$.

We consider the weak factorization on $\mathrm{Mod}(T)$ cofibrantly generated by the map $Y(A) \to Y(B)$ for $B \twoheadrightarrow A$ a display map.

We call the left class "cofibrations" and the right class "Anodyne fibrations".

We see $\mathrm{Mod}(T)$ as a premodel category, with anodyne cofibration being the isomorphisms and all maps being fibrations.

In particular, the Yoneda embedding define a fully faithful functor $Y : C_T^{op} \to \mathrm{Mod}(T) \subset \mathrm{Fun}(C_T, \mathrm{Set})$.

We consider the weak factorization on $\mathrm{Mod}(T)$ cofibrantly generated by the map $Y(A) \to Y(B)$ for $B \twoheadrightarrow A$ a display map.

We call the left class "cofibrations" and the right class "Anodyne fibrations".

We see $\mathrm{Mod}(T)$ as a premodel category, with anodyne cofibration being the isomorphisms and all maps being fibrations.

The category $\mathrm{Cof}_\omega(\mathrm{Mod}(T))$ of finitely presentable cofibrant objects in $\mathrm{Mod}(T)$ is equivalent to (the Cauchy completion) C_T^{op}.

We get a (left semi) model structure $s\text{Mod}(T)$ using simplicial completion:

We get a (left semi) model structure $s\mathrm{Mod}(T)$ using simplicial completion:

1. The underlying category is the category of simplicial models.

We get a (left semi) model structure $s\mathrm{Mod}(T)$ using simplicial completion:

1. The underlying category is the category of simplicial models.
2. The fibrant objects are models where each dependent is interpreted as a Kan fibration (between Kan complexes).

We get a (left semi) model structure $s\mathrm{Mod}(T)$ using simplicial completion:

1. The underlying category is the category of simplicial models.
2. The fibrant objects are models where each dependent is interpreted as a Kan fibration (between Kan complexes).
3. Equivalently, fibrant objects are the simplicial models $C_T \to \mathrm{sSet}$ which takes values in Kan complexes and send each display map to a Kan fibration.

We get a (left semi) model structure $s\mathrm{Mod}(T)$ using simplicial completion:

1. The underlying category is the category of simplicial models.
2. The fibrant objects are models where each dependent is interpreted as a Kan fibration (between Kan complexes).
3. Equivalently, fibrant objects are the simplicial models $C_T \to \mathrm{sSet}$ which takes values in Kan complexes and send each display map to a Kan fibration. (So all pullbacks appearing in the definition of pullbacks are pullback of fibrations)

We get a (left semi) model structure $s\text{Mod}(T)$ using simplicial completion:

1. The underlying category is the category of simplicial models.
2. The fibrant objects are models where each dependent is interpreted as a Kan fibration (between Kan complexes).
3. Equivalently, fibrant objects are the simplicial models $C_T \to \text{sSet}$ which takes values in Kan complexes and send each display map to a Kan fibration. (So all pullbacks appearing in the definition of pullbacks are pullback of fibrations)
4. Weak equivalence between fibrant objects are the levelwise weak equivalences.

We can apply our main theorem to this premodel category $\mathrm{Mod}(T)$. We obtain:

Theorem (H. - L. S.)

The localization of the model category $s\mathrm{Mod}(T)$ of strict simplicial model of T at weak equivalence is equivalent to the ∞-category of weak models of T.

We can apply our main theorem to this premodel category Mod(T). We obtain:

Theorem (H. - L. S.)

The localization of the model category sMod(T) of strict simplicial model of T at weak equivalence is equivalent to the ∞-category of weak models of T.

Where a "weak model" is a (pseudo) functor $C_T \to$ Spaces that preserve the terminal objects sends pullback of display map to (homotopy) pullbacks.

When T is a Lawvere theory (all type are non-dependant), we recover a Theorem of Badzioch: Any weak model of A Lawvere theory is homotopically equivalent to a Kan complex with a strict T-model structure. For example:

When T is a Lawvere theory (all type are non-dependant), we recover a Theorem of Badzioch: Any weak model of A Lawvere theory is homotopically equivalent to a Kan complex with a strict T-model structure. For example:

Example

The ∞-category of simplicial abelian groups is equivalent to the ∞-category of (weak) model of the Lawvere theory of Abelian groups.

When T is a Lawvere theory (all type are non-dependant), we recover a Theorem of Badzioch: Any weak model of A Lawvere theory is homotopically equivalent to a Kan complex with a strict T-model structure. For example:

Example

The ∞-category of simplicial abelian groups is equivalent to the ∞-category of (weak) model of the Lawvere theory of Abelian groups. Note that these are not the same as the (weak)

models of the commutative operads (i.e. E_∞-algebras).

Example

Taking T the theory of category mentioned before $s\mathrm{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.

Example

Taking T the theory of category mentioned before $s\mathrm{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.
bigskip
This model structure is Quillen equivalent to the one for (non-complete) Segal spaces,

Example

Taking T the theory of category mentioned before $s\mathrm{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.
bigskip
This model structure is Quillen equivalent to the one for (non-complete) Segal spaces, but that does not follow immediately from our theorem.

Example

Taking T the theory of category mentioned before $s\mathrm{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.
bigskip
This model structure is Quillen equivalent to the one for (non-complete) Segal spaces, but that does not follow immediately from our theorem.

The previous theorem say that it is equivalent to a certain category of functor $C_T \to \mathrm{Space}$, but $C_T^{\mathrm{op}} \neq \Delta$.

Example

Taking T the theory of category mentioned before $s\mathrm{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.
bigskip
This model structure is Quillen equivalent to the one for (non-complete) Segal spaces, but that does not follow immediately from our theorem.

The previous theorem say that it is equivalent to a certain category of functor $C_T \to \mathrm{Space}$, but $C_T^{op} \neq \Delta$.

C_T^{op} is the full subcategory of Cat of categories that are free on graphs (so it contains Δ).

Example

Taking T the theory of category mentioned before $s\mathrm{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.
bigskip
This model structure is Quillen equivalent to the one for (non-complete) Segal spaces, but that does not follow immediately from our theorem.

The previous theorem say that it is equivalent to a certain category of functor $C_T \to \mathrm{Space}$, but $C_T^{\mathrm{op}} \neq \Delta$.

C_T^{op} is the full subcategory of Cat of categories that are free on graphs (so it contains Δ). And the functors we consider are the one that satisfies an analogue of the Segal conditions

Example

Taking T the theory of category mentioned before $s\text{Mod}(T)$ is a Model structure on simplicial categories whose fibrant objects are simplicial categories C where the simplicial set of objects C_0 is a Kan complex and the map $C_1 \to C_0 \times C_0$ is a Kan fibration.
bigskip
This model structure is Quillen equivalent to the one for (non-complete) Segal spaces, but that does not follow immediately from our theorem.

The previous theorem say that it is equivalent to a certain category of functor $C_T \to \text{Space}$, but $C_T^{op} \neq \Delta$.

C_T^{op} is the full subcategory of Cat of categories that are free on graphs (so it contains Δ). And the functors we consider are the one that satisfies an analogue of the Segal conditions (their value on any graph is recovered as a limit of $F(\Delta[0])$ and $F(\Delta[1])$ over all verticies and edge of the graph).

In general, the theorem tell you in ∞-categorical language what the the homotopy theory of strict model actually is.

In general, the theorem tell you in ∞-categorical language what the the homotopy theory of strict model actually is.

But generally not in a convenient way.

In general, the theorem tell you in ∞-categorical language what the the homotopy theory of strict model actually is.

But generally not in a convenient way. The hard part is to compare the notion of “weak model” produced by the theorem with the kind of weak structure we want.

To solve this sort of problem, we have a more precise version of the theorem,

To solve this sort of problem, we have a more precise version of the theorem, where we can input a proposed notion of weak models.

To solve this sort of problem, we have a more precise version of the theorem, where we can input a proposed notion of weak models.

Let C be a κ-combinatorial premodel category and $D \subset \mathrm{Cof}_\kappa(C)$ a full subcategory. We have a Realization-Nerve adjunction:

$$R : \widehat{D} \leftrightarrows C : N_D$$

To solve this sort of problem, we have a more precise version of the theorem, where we can input a proposed notion of weak models.

Let C be a κ-combinatorial premodel category and $D \subset \mathrm{Cof}_\kappa(C)$ a full subcategory. We have a Realization-Nerve adjunction:

$$R : \widehat{D} \leftrightarrows C : N_D$$

which extend to a simplicial Quillen adjunction

$$R : s\widehat{D}_{\mathrm{Proj}} \leftrightarrows sC : N_D$$

To solve this sort of problem, we have a more precise version of the theorem, where we can input a proposed notion of weak models.

Let C be a κ-combinatorial premodel category and $D \subset \mathrm{Cof}_{\kappa}(C)$ a full subcategory. We have a Realization-Nerve adjunction:

$$R : \widehat{D} \leftrightarrows C : N_D$$

which extend to a simplicial Quillen adjunction

$$R : s\widehat{D}_{\mathrm{Proj}} \leftrightarrows sC : N_D$$

We want "weak models" to be modeled by a specific Bousfield localization of $s\widehat{D}_{\mathrm{Proj}}$, and figure out nice conditions under which strict and weak model are equivalent.

To solve this sort of problem, we have a more precise version of the theorem, where we can input a proposed notion of weak models.

Let C be a κ-combinatorial premodel category and $D \subset \mathrm{Cof}_\kappa(C)$ a full subcategory. We have a Realization-Nerve adjunction:

$$R : \widehat{D} \leftrightarrows C : N_D$$

which extend to a simplicial Quillen adjunction

$$R : s\widehat{D}_{\mathrm{Proj}} \leftrightarrows sC : N_D$$

We want "weak models" to be modeled by a specific Bousfield localization of $s\widehat{D}_{\mathrm{Proj}}$, and figure out nice conditions under which strict and weak model are equivalent.

Example

Take $C = \mathrm{Mod}(\mathrm{Cat})$ and $D = \Delta$ (seen as a dense full subcategory of categories). And the localization $s\widehat{\Delta}^{Segal}_{Proj}$.

We also assume that C is κ-combinatorial and that it has a set of generating cofibration in D.

Theorem (H. - L. S.)

In the situation above, Then given S a set of maps in $s\widehat{D}$, the functors R and N_D induces a Quillen equivalence between sC and the localization $s\widehat{D}^{S\text{-}Loc}_{Prof}$ if and only if:

- *All maps in S are sent to ismorphism by the derived functor $\overline{R} : \widehat{D} \to Ho(sC)$. (Equivalently, each fibrant object of sC is sent by N_D to an S-local object).*

We also assume that C is κ-combinatorial and that it has a set of generating cofibration in D.

Theorem (H. - L. S.)

In the situation above, Then given S a set of maps in $s\widehat{D}$, the functors R and N_D induces a Quillen equivalence between sC and the localization $s\widehat{D}^{S\text{-}Loc}_{Prof}$ if and only if:

- *All maps in S are sent to ismorphism by the derived functor $\overline{R} : \widehat{D} \to Ho(sC)$. (Equivalently, each fibrant object of sC is sent by N_D to an S-local object).*
- *The restriction of N_D to $Cof_\kappa(C)$:*
    1. *Sends pushout of generating cofibrations to homotopy pushouts in $s\widehat{D}^{S\text{-}Loc}_{Prof}$.*

We also assume that C is κ-combinatorial and that it has a set of generating cofibration in D.

Theorem (H. - L. S.)

In the situation above, Then given S a set of maps in $s\widehat{D}$, the functors R and N_D induces a Quillen equivalence between sC and the localization $s\widehat{D}^{S\text{-Loc}}_{Prof}$ if and only if:

- *All maps in S are sent to ismorphism by the derived functor $\overline{R} : \widehat{D} \to Ho(sC)$. (Equivalently, each fibrant object of sC is sent by N_D to an S-local object).*
- *The restriction of N_D to $Cof_\kappa(C)$:*
  1. *Sends pushout of generating cofibrations to homotopy pushouts in $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*
  2. *Sends the initial object to a homotopically initial objects of $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*

We also assume that C is κ-combinatorial and that it has a set of generating cofibration in D.

Theorem (H. - L. S.)

In the situation above, Then given S a set of maps in $s\widehat{D}$, the functors R and N_D induces a Quillen equivalence between sC and the localization $s\widehat{D}^{S\text{-Loc}}_{Prof}$ if and only if:

- *All maps in S are sent to ismorphism by the derived functor $\overline{R} : \widehat{D} \to Ho(sC)$. (Equivalently, each fibrant object of sC is sent by N_D to an S-local object).*
- *The restriction of N_D to $Cof_\kappa(C)$:*
  1. *Sends pushout of generating cofibrations to homotopy pushouts in $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*
  2. *Sends the initial object to a homotopically initial objects of $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*
  3. *Sends κ-small transfinite to homotopy colimits.*

We also assume that C is κ-combinatorial and that it has a set of generating cofibration in D.

Theorem (H. - L. S.)

In the situation above, Then given S a set of maps in $s\widehat{D}$, the functors R and N_D induces a Quillen equivalence between sC and the localization $s\widehat{D}^{S\text{-Loc}}_{Prof}$ if and only if:

- *All maps in S are sent to ismorphism by the derived functor $\overline{R} : \widehat{D} \to Ho(sC)$. (Equivalently, each fibrant object of sC is sent by N_D to an S-local object).*
- *The restriction of N_D to $Cof_\kappa(C)$:*
  1. *Sends pushout of generating cofibrations to homotopy pushouts in $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*
  2. *Sends the initial object to a homotopically initial objects of $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*
  3. *Sends κ-small transfinite to homotopy colimits.*
  4. *Sends the generating anodyne cofibrations of sC to weak equivalences in $s\widehat{D}^{S\text{-Loc}}_{Prof}$.*

We also assume that C is κ-combinatorial and that it has a set of generating cofibration in D.

Theorem (H. - L. S.)

In the situation above, Then given S a set of maps in $s\widehat{D}$, the functors R and N_D induces a Quillen equivalence between sC and the localization $s\widehat{D}^{S\text{-}Loc}_{Prof}$ if and only if:

- *All maps in S are sent to ismorphism by the derived functor $\overline{R} : \widehat{D} \to Ho(sC)$. (Equivalently, each fibrant object of sC is sent by N_D to an S-local object).*
- *The restriction of N_D to $Cof_\kappa(C)$:*
  1. *Sends pushout of generating cofibrations to homotopy pushouts in $s\widehat{D}^{S\text{-}Loc}_{Prof}$.*
  2. *Sends the initial object to a homotopically initial objects of $s\widehat{D}^{S\text{-}Loc}_{Prof}$.*
  3. *Sends κ-small transfinite to homotopy colimits.*
  4. *Sends the generating anodyne cofibrations of sC to weak equivalences in $s\widehat{D}^{S\text{-}Loc}_{Prof}$.*

Comming back to our example of categories:

Example

By the previous theorem, in order to show that the model category $s\text{Mod}(\text{Cat})$ is equivalent to the one for Segal spaces,

Comming back to our example of categories:

Example

By the previous theorem, in order to show that the model category $s\mathrm{Mod}(\mathrm{Cat})$ is equivalent to the one for Segal spaces, we only need to show that given a graph G, then the pushout corresponding to add an arrow (or an object) to the free category on G is a homotopy pushout in the model category of Segal spaces.
This is not trivial,

Comming back to our example of categories:

Example

By the previous theorem, in order to show that the model category $s\text{Mod}(\text{Cat})$ is equivalent to the one for Segal spaces, we only need to show that given a graph G, then the pushout corresponding to add an arrow (or an object) to the free category on G is a homotopy pushout in the model category of Segal spaces.
This is not trivial, but is well known.

Example

Simpson's Semi-strictification conjecture says that one can represent weak (∞, n)-categories using "semi-strict" (∞, n)-category where both associativity and exchange law are strict, but unit law are weak.

Example

Simpson's Semi-strictification conjecture says that one can represent weak (∞, n)-categories using "semi-strict" (∞, n)-category where both associativity and exchange law are strict, but unit law are weak. Using the above, we can phrase a model independant version of the conjecture:

Example

Simpson's Semi-strictification conjecture says that one can represent weak (∞, n)-categories using "semi-strict" (∞, n)-category where both associativity and exchange law are strict, but unit law are weak. Using the above, we can phrase a model independant version of the conjecture:

Conjecture

For any (strict) "positive" polygraphs/computads X (that is where each generators is glued on cell that are not identities) the pushouts corresponding to adding a new cell to X is a homotopy pushout of weak (∞, ∞)-categories.

Example

Simpson's Semi-strictification conjecture says that one can represent weak (∞, n)-categories using "semi-strict" (∞, n)-category where both associativity and exchange law are strict, but unit law are weak. Using the above, we can phrase a model independant version of the conjecture:

Conjecture

For any (strict) "positive" polygraphs/computads X (that is where each generators is glued on cell that are not identities) the pushouts corresponding to adding a new cell to X is a homotopy pushout of weak (∞, ∞)-categories.

A version of this in the special case of "pasting diagrams" (instead of all positive polygraphs) has been established last year by Tim Campion.

I'm concluding with one more example of the first theorem which I think is of interest,

I'm concluding with one more example of the first theorem which I think is of interest, or maybe just confusing.

I'm concluding with one more example of the first theorem which I think is of interest, or maybe just confusing.

Example

Take C to be the (1-)category of strict n-categories.

I'm concluding with one more example of the first theorem which I think is of interest, or maybe just confusing.

Example

Take C to be the (1-)category of strict n-categories.

Inside of C we have Joyal "Disks" category $\Theta_n \subset C$.

I'm concluding with one more example of the first theorem which I think is of interest, or maybe just confusing.

Example

Take C to be the (1-)category of strict n-categories.

Inside of C we have Joyal "Disks" category $\Theta_n \subset C$. We Consider the premodel structure C cofibrantly generated by the map $\varnothing \to X$ for $X \in \Theta_n$ (and no anodyne cofibrations).

I'm concluding with one more example of the first theorem which I think is of interest, or maybe just confusing.

Example

Take C to be the (1-)category of strict n-categories.

Inside of C we have Joyal "Disks" category $\Theta_n \subset C$. We Consider the premodel structure C cofibrantly generated by the map $\varnothing \to X$ for $X \in \Theta_n$ (and no anodyne cofibrations).

Then the simplicial completion sC is equivalent to the category of Θ_n-spaces.

I'm concluding with one more example of the first theorem which I think is of interest, or maybe just confusing.

Example

Take C to be the (1-)category of strict n-categories.

Inside of C we have Joyal "Disks" category $\Theta_n \subset C$. We Consider the premodel structure C cofibrantly generated by the map $\varnothing \to X$ for $X \in \Theta_n$ (and no anodyne cofibrations).

Then the simplicial completion sC is equivalent to the category of Θ_n-spaces.

In particular, there is a Bousfield localization of sC (i.e. strict simplicial n-categories) which models weak (∞, n)-categories. The case $n = 1$ is exactly Horel's model structure on simplicial categories.

All of this works more generally when C is an extensive category and Θ is replaced by any set of connected objects.