Clans and finite direct categories

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Part I : Functorial Semantics

Algebraic theories

Algebraic theories are given given by **sorts**, generators **generators** and **equations**.

- Theory $\mathbb{T}_{\mathsf{Mon}}$ of monoids:

$$
\begin{aligned}
&\vdash M \\
&\vdash e : M \\
x\,y : M &\vdash x \cdot y : M \\
x : M &\vdash e \cdot x = x \cdot e = x \\
x\,y\,z : M &\vdash (x \cdot y) \cdot z = x \cdot (y \cdot z)
\end{aligned}
$$

- Theory $\mathbb{T}_{\mathsf{Gph}}$ of graphs

$$
\begin{aligned}
&\vdash V \\
&\vdash E \\
x : E &\vdash s(x) : V \\
x : E &\vdash t(x) : V
\end{aligned}
$$

Syntactic category

Definition

The syntactic category $\mathcal{C}[\mathbb{T}]$ of an algebraic theory $\mathbb{T}$ is given as follows:

- **Objects** are **contexts**, i.e. lists $(x_1 : S_1, \ldots x_n : S_n)$ of **sorted variables**.
- **Morphisms** from $(x_1{:}S_1, \ldots, x_n{:}S_n)$ to $(y_1{:}T_1, \ldots, y_k{:}T_k)$ are k-tuples of equivalence classes of terms in variables $x_1, \ldots, x_n$ modulo equations.
- **Composition** is given by subsitution.

Theorem (Lawvere)

$\mathcal{C}[\mathbb{T}]$ has finite products (given by concatenation), and the models of $\mathbb{T}$ (in Set) correspond to finite-product preserving functors from $\mathcal{C}[\mathbb{T}]$ to Set:

$$\mathsf{Mod}(\mathbb{T}) \simeq \mathbf{FP}(\mathcal{C}[\mathbb{T}], \mathsf{Set})$$

Lawvere theories and finite-product theories

If $\mathbb{T}$ is **single-sorted** (e.g. $\mathbb{T}_{\mathsf{Mon}}$) then contexts are fully determined by their length (up to renaming of variables), and the objects of $\mathcal{C}[\mathbb{T}]$ are finite powers of a single generating object. In this case we speak of a **Lawvere theory**.

More generally we define:

Definition

- A **finite-product theory** is a small category $\mathcal{C}$ with finite products.
- A **model** of a finite-product theory $\mathcal{C}$ is a finite-product preserving functor $\mathcal{C} \to \mathsf{Set}$.

$$\mathsf{Mod}(\mathcal{C}) := \mathbf{FP}(\mathcal{C}, \mathsf{Set}) \overset{\text{full}}{\subseteq} [\mathcal{C}, \mathsf{Set}]$$

This makes sense since one can show that every finite-product category $\mathcal{C}$ is equivalent to the syntactic category of an algebraic theory (possibly with infinitely many generators and relations).

Every finite-product theory contravariantly embeds into its models:

$$\begin{array}{ccc} & & \mathcal{C}^{\mathsf{op}} \\ & \overset{H}{\swarrow} & \downarrow \text{よ} \\ \mathsf{Mod}(\mathcal{C}) & \subseteq & [\mathcal{C}, \mathsf{Set}] \end{array}$$

Finite-limit theories

There are classes of 'essentially' algebraic structures that cannot be represented as models of finite-product theories, most notably **categories**.
The problem is that composition in a category is not defined for arbitary pairs of arrows: the set of composable pairs is not given by a product but by a **pullback**! This motivates the following:

Definition

- A **finite-limit theory** is a small category with finite limits
- a **model** of a finite-limit theory is a finite-limit preserving functor $A : \mathcal{C} \to \mathsf{Set}$

Like finite-product theories, finite-limit theories embed contravariantly into their models

$$\begin{array}{ccc} & & \mathcal{C}^{\mathsf{op}} \\ & \overset{Z}{\swarrow} & \downarrow \text{よ} \\ \mathsf{Mod}(\mathcal{C}) & \subseteq & [\mathcal{C}, \mathsf{Set}] \end{array} \quad ,$$

and in this case, the essential image of Z admits a straightforward generalizations!

Duality for finite-limit theories (Gabriel-Ulmer duality[1])

Proposition

Let $\mathcal{C}$ be a finite-limit theory.

1. A model $A \in \mathsf{Mod}(\mathcal{C})$ is representable by an object of $\mathcal{C}$ iff it is **compact**, i.e. $\mathsf{Mod}(\mathcal{C})(A, -)$ preserves filtered colimits.
2. The category $\mathsf{Mod}(\mathcal{C}) = \mathbf{FP}(\mathcal{C}, \mathsf{Set})$ is **locally finitely presentable**, i.e. cocomplete with a dense set of compact objects.

Theorem

There is a contravariant bi-equivalence of 2-categories

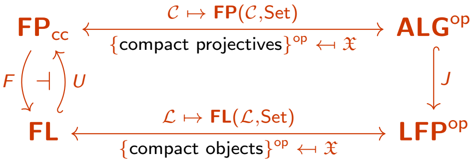
$$\mathbf{FL} \underset{\mathcal{L} \mapsto \mathsf{Mod}(\mathcal{L}) := \mathbf{FL}(\mathcal{L}, \mathsf{Set})}{\overset{\{\text{compact objects}\}^{\mathrm{op}} \leftarrow\!\shortmid \mathfrak{X}}{\longleftrightarrow}} \mathbf{LFP}^{\mathrm{op}}.$$

between the 2-category **FL** of small finite-limit theories, and the 2-category **LFP** of locally finitely presentable categories.

[1] P. Gabriel and F. Ulmer (1971). *Lokal präsentierbare Kategorien*. Springer-Verlag.

Duality for finite-product theories[2]

There's a 'restriction' of G–U duality to **finite-product theories** (corresponding to many-sorted **ordinary algebraic theories**):



- $\mathbf{FP}_{cc}$ is the 2-category of Cauchy-complete finite-product categories
- **ALG** is the 2-category of **algebraic categories** and **algebraic functors**
  - An **algebraic category** is an l.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
  - An **algebraic functor** is a functor that preserves small limits, filtered colimits, and regular epimorphisms.

  sifted colimits
- Clan-duality can be viewed as a **refinement** of GU-duality which allows to control the amount of limit-preservation in the models

[2] J. Adámek, J. Rosický, and E.M. Vitale (2010). *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press.

GATs and Clans

Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
  - Freyd's **essentially algebraic theories**[3]
  - Cartmell's **generalized algebraic theories**[4] (or 'dependent algebraic theories')
  - Johnstone's **cartesian theories**[5]
  - Palmgren and Vickers' **quasi-equational theories**[6]
  - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They're as expressive as FL-theories, but 'finer', i.e. closer to syntax

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[3] P. Freyd (1972). "Aspects of topoi". *Bulletin of the Australian Mathematical Society*.
[4] J. Cartmell (1986). "Generalised algebraic theories and contextual categories". *Annals of Pure and Applied Logic*.
[5] P.T. Johnstone (2002). *Sketches of an elephant: a topos theory compendium. Vol. 2*. Oxford: Oxford University Press.
[6] E. Palmgren and S. J. Vickers (2007). "Partial horn logic and Cartesian categories". *Annals of Pure and Applied Logic*.

Generalized Algebraic Theories

John Cartmell

Generalized Algebraic Theories[7] can have 'sort dependencies'. For example:

The GAT of families of pointed sets:

$$\begin{aligned} &\vdash A \\ x : A &\vdash B(x) \\ x : A &\vdash p(x) : B(x) \end{aligned}$$

The GAT rGph^* of reflexive graphs:

$$\begin{aligned} &\vdash V \\ x\, y : V &\vdash E(x,y) \\ x : V &\vdash r(x) : E(x,x) \end{aligned}$$

The GAT $\mathbb{T}_{\mathsf{Cat}}$ of categories:

$$\begin{aligned} &\vdash O \\ x\, y : O &\vdash A(x,y) \\ x : O &\vdash \mathrm{id}(x) : A(x,x) \\ x\, y\, z : O\,,\ f : A(x,y)\,,\ g : A(y,z) &\vdash g \circ f : A(x,z) \\ x\, y : O\,,\ f : A(x,y) &\vdash \mathrm{id}(y) \circ f = f \\ x\, y : O\,,\ f : A(x,y) &\vdash f \circ \mathrm{id}(x) = f \\ w\, x\, y\, z : O\,,\ e : A(w,x)\,, & \\ f : A(x,y)\,,\ g : A(y,z) &\vdash (g \circ f) \circ e = g \circ (f \circ e) \end{aligned}$$

[7] J. Cartmell (1978). "Generalised algebraic theories and contextual categories". available at https://ncatlab.org/nlab/files/Cartmell-Thesis.pdf. PhD thesis. Oxford University
J. Cartmell (1986). "Generalised algebraic theories and contextual categories". *Annals of Pure and Applied Logic*

The GAT of semisimplicial sets/ojects

$$
\begin{aligned}
&\vdash A_0\\
x_0\, x_1 : A_0 &\vdash A_1(x_0, x_1)\\
x_0\, x_1\, x_2 : A_0\,,\, x_{01} : A_1(x_0, x_1)\,,\, x_{02} : A_1(x_0, x_2)\,,\, x_{12} : A_1(x_1, x_2) &\vdash A_2(x_{01}, x_{02}, x_{12})\\
\dots &\vdash \dots
\end{aligned}
$$

More generally, we can write such a GAT consisting only of type declaration for any **direct locally finte category**.

Definition

a category $\mathbb{D}$ is called **direct**, if it does not admit an infinite inverse chain

$$A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \dots$$

of non-identity arrows.

Syntactic category of a GAT

GATs also have a syntactic category $\mathbb{C}[\mathbb{T}]$ of contexts and substitutions, but now contexts may be dependent:

$$(x_1{:}S_1, x_2{:}S_2(x_1), \ldots, x_n{:}S(x_1, \ldots, x_{n-1}))$$

and similarly for substitutions.

Syntactic categories of GATs are **contextual categories** (Cartmell), and in particular **clans**:

Definition (Taylor 1987, Joyal 2017)

Clan: small category $\mathcal{T}$ with 1, and class $\mathcal{T}_\dagger \subseteq \mathrm{mor}(\mathcal{T})$ of '**display maps**' (written $\twoheadrightarrow$) such that

1. pullbacks of display maps along all maps exist and are display maps $\begin{array}{ccc} \Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\ q \downarrow & \lrcorner & \downarrow p \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$,
2. display maps are closed under composition, and
3. terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.

The display maps in $\mathbb{C}[\mathbb{T}]$ are **dependent projections**

$$(x_1{:}S_1, x_2{:}S_2(x_1), \ldots, x_n{:}S(x_1, \ldots, x_{n-1}) \ldots, x_{n+k}{:}S(x_1, \ldots, x_{n+k-1}))$$
$$\twoheadrightarrow (x_1{:}S_1, x_2{:}S_2(x_1), \ldots, x_n{:}S(x_1, \ldots, x_{n-1}))$$

from longer to shorter contexts (closed under isos).

Examples

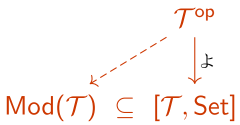
- Finite-product theories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}_\dagger = \{\text{product projections}\}$ (*'FP-clans'*)
- Finite-limit theories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}_\dagger = \mathsf{mor}(\mathcal{L})$ (*'FL-clans'*)
- For $\mathbb{D}$ direct locally finite, its finite cocompletion $[\mathbb{D}^{\mathsf{op}}, \mathsf{Fin}]$ is a **coclan**, with **monomorphisms** as **co-display maps**[8].

[8] This is stated in terms of contextual categories in : C. Leena Subramaniam (Oct. 2021). "From dependent type theory to higher algebraic structures". and also mentioned in M. Makkai (1995). "First order logic with dependent sorts, with applications to category theory". *Preprint*.

Models

> ***Definition***
>
> A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathsf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathsf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathsf{Set}]$ of models is l.f.p. and contains $\mathcal{T}^{\mathsf{op}}$.



- For FP-clans $(\mathcal{C}, \mathcal{C}_\dagger)$ we have $\mathsf{Mod}(\mathcal{C}, \mathcal{C}_\dagger) = \mathbf{FP}(\mathcal{C}, \mathsf{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}_\dagger)$ we have $\mathsf{Mod}(\mathcal{L}, \mathcal{L}_\dagger) = \mathbf{FL}(\mathcal{L}, \mathsf{Set})$.
- For clans $\mathcal{C}_{\mathbb{D}} = [\mathbb{D}^{\mathsf{op}}, \mathsf{Fin}]^{\mathsf{op}}$ arising from direct locally finite categories, we have $\mathsf{Mod}(\mathcal{C}_{\mathbb{D}}) = [\mathbb{D}^{\mathsf{op}}, \mathsf{Set}]$.

The clan of categories

- The syntactic category $\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]$ of the GAT $\mathbb{T}_{\mathsf{Cat}}$ has contexts

$$(x_1 \dots x_n : O, f_1 : A(x_{i_1}, x_{j_1}), \dots f_k : A(x_{i_k}, x_{j_k}))$$

  as objects, and substitutions as morphisms.
- As for any clan, we have the Yoneda embedding

$$\text{よ} \; : \; \mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]^{\mathsf{op}} \longrightarrow \mathsf{Mod}(\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]) \simeq \mathsf{Cat}.$$

- Its image is the full subcategory of Cat on **free categories on finite graphs**.
- Display maps correspond (contravariantly) to **graph inclusions**
- Compare: the **finite-limit theory** $\mathcal{L}_{\mathsf{Cat}}$ of categories is identified by Yoneda with the larger subcategory of **finitely presentable** (compact) categories.

$$\mathcal{L}_{\mathsf{Cat}}^{\mathsf{op}} \xrightarrow{\cong} \{\text{f.p. categories}\} \subseteq \mathsf{Cat}$$

Towards duality for clans

- Note that the different clans can have the same category of Set-models
- For example, algebraic theories give rise to clans either as finite-product theories or as finite-limit theories
- To get a duality theory for clans, have to **refine** Gabriel–Ulmer duality.
- We do this by equipping the categories of models with additional data in form of a **weak factorization system**

The extension–full weak factorization system

Definition

Let $\mathcal{T}$ be a clan and $よ : \mathcal{T}^{\mathsf{op}} \to \mathsf{Mod}(\mathcal{T})$. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathsf{Mod}(\mathcal{T})$:

$$\mathcal{F} = \mathbf{RLP}(よ(\mathcal{T}^{\dagger})) \qquad \textbf{'full maps'}$$

$$\mathcal{E} = \mathbf{LLP}(\mathcal{F}) \qquad \textbf{'extensions'}$$

Call $A \in \mathsf{Mod}(\mathcal{T})$ a 0-**extension**, if $(0 \to A) \in \mathcal{E}$.

- Representable models $よ(\Gamma) = \mathcal{T}(\Gamma, -)$ are 0-extensions since all $\Gamma \twoheadrightarrow 1$ are display maps.
- The same weak factorization system was also introduced by Henry[9], who called coclans **cofibration categories**.
- In the case of direct locally finite categories $\mathbb{D}$, the w.f.s. on $[\mathbb{D}^{\mathsf{op}}, \mathsf{Set}]$ is cofibrantly generated by **boundary inclusions** $\partial d \hookrightarrow y(d)$, and Makkai[10] uses the term **very surjective** for the right class.
- For general clans, the full maps are those whose **display-naturality squares** are **weak pullbacks**, in particular full maps are always regular epimorphisms.

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[9] S. Henry (2016). "Algebraic models of homotopy types and the homotopy hypothesis". *arXiv:1609.04622*.

[10] M. Makkai (1995). "First order logic with dependent sorts, with applications to category theory". *Preprint*.

Examples

- If $\mathcal{T}$ is a FL-clan, then
  - only isos are full in $\mathsf{Mod}(\mathcal{T})$, and
  - all maps are extensions.
- If $\mathcal{T}$ is a FP-clan, then
  - $\mathsf{Mod}(\mathcal{T})$ is Barr-exact,
  - the full maps are the **regular epis**, and
  - the 0-extensions are the **projective objects**.
- In $\mathsf{Cat} = \mathsf{Mod}(\mathbb{T}_{\mathsf{Cat}})$:
  - full maps are functors that are **full and surjective on objects**,
  - and 0-extensions are **free categories**.

Duality for clans

Theorem (F)

There is a contravariant bi-equivalence of 2-categories

$$\mathbf{Clan}_{\mathsf{cc}} \quad \xleftrightarrow[\mathcal{T} \mapsto \mathsf{Mod}(\mathcal{T})]{\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}} \leftarrow\!\shortmid\, \mathfrak{X}} \quad \mathbf{cAlg}^{\mathsf{op}}$$

where

- $\mathbf{Clan}_{\mathsf{cc}}$ is the 2-category of **Cauchy-complete**[11] clans,
- $\mathbf{cAlg}$ is the 2-category of **clan-algebraic categories**, i.e. l.f.p. categories $\mathfrak{X}$ equipped with an 'extension/full' WFS $(\mathcal{E}, \mathcal{F})$ such that
  1. the **full subcategory** $\mathsf{CZE}(\mathfrak{X}) \subseteq \mathfrak{X}$ **on compact** 0**-extensions** is dense in $\mathfrak{X}$,
  2. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps in $\mathsf{CZE}(\mathfrak{X})$, and
  3. $\mathfrak{X}$ has **full and effective quotients of componentwise-full equivalence relations**.

As special cases for FL-clans and FP-clans we recover

- Gabriel–Ulmer duality, and
- Adamek–Rosicky–Vitale's characterization of **algebraic categories** as Barr-exact LFP categories which are generated by compact projectives[12].

[11] A clan $\mathcal{T}$ is Cauchy-complete if idempotents split in $\mathcal{T}$, and retracts of display maps are display maps.

[12] Theorem 9.15 in J. Adámek, J. Rosický, and E.M. Vitale (2010). *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press

Proof sketch

- Have to show that:
  1. $\mathsf{Mod}(\mathcal{T})$ is clan-algebraic for all clans $\mathcal{T}$.
  2. $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}$ is a clan for all clan-algebraic categories $\mathfrak{X}$ (with extensions as display maps).
  3. $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ for all clan-algebraic categories $\mathfrak{X}$.
  4. $\mathcal{T} \simeq \mathsf{CZE}(\mathsf{Mod}(\mathcal{T}))^{\mathsf{op}}$ for all Cauchy-complete clans $\mathcal{T}$.
- 1 and 2 are easy
- For 3 we use a Reedy factorization on 2-truncated semi-simplicial models
- For 4 we use the **fat small object argument**[13], which implies that 0-extensions are filtered colimits of representable algebras.

[13] M. Makkai, J. Rosicky, and L. Vokrinek (2014). "On a fat small object argument". *Advances in Mathematics*.

Part II : Models in Higher Types

Models in higher types

- Straightforward notion of **higher model**:A higher model of $\mathcal{T}$ is an ∞-functor $\mathcal{T} \to \mathcal{S}$ into the ∞-category $\mathcal{S}$ of **homotopy types**, which preserves 1 and display pullbacks in the ∞-categorical sense (here $\mathcal{T}$ is viewed as ∞-category via its nerve).
- Simon told us yesterday that the higher models of the clan $\mathcal{T}_{\mathsf{Cat}}$ of categories are **Segal spaces**.
- However, there is more than one clan whose category of 1-models is Cat!
- We can find other clans by exhibiting **clan-algebraic factorization systems** on Cat.

Four clan-algebraic weak factorization systems on Cat

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$\mathcal{F}_1 = \{\text{full and surjective-on-objects functors}\}$$
$$\mathcal{F}_2 = \{\text{full and bijective-on-objects functors}\}$$
$$\mathcal{F}_3 = \{\text{fully faithful and surjective-on-objects functors}\}$$
$$\mathcal{F}_4 = \{\text{isos}\}$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on Cat.

Four clans for categories

These correspond to the following clans:

$\mathcal{T}_1 = \{$free cats on fin. graphs$\}^{\mathsf{op}}$

$\mathcal{T}_2 = \{$free cats on fin. graphs$\}^{\mathsf{op}}$

$\mathcal{T}_3 = \{$f.p. cats$\}^{\mathsf{op}}$

$\mathcal{T}_4 = \{$f.p. cats$\}^{\mathsf{op}}$

$\mathcal{T}_1^{\dagger} = \{$graph inclusions$\}$

$\mathcal{T}_2^{\dagger} = \{$injective-on-edges maps$\}$

$\mathcal{T}_3^{\dagger} = \{$injective-on-objects functors$\}$

$\mathcal{T}_4^{\dagger} = \{$all functors$\}$

Syntax: four GATs for categories

- Syntactically, adding $(2 \to 1)$ to the generators turns the diagonal of the type $\vdash\ O$ of objects into a display map. This corresponds to adding an extensional identity type with rules
  - $x\,y : O\ \vdash\ E(x,y)$ type
  - $x : O\ \vdash\ r : E(x,x)$
  - $x\,y : O\,,\ p : E(x,y)\ \vdash\ x = y$
  - $x\,y : O\,,\ p\,q : E(x,y)\ \vdash\ p = q$

  to the GAT.
- Similarly, adding $(\mathbb{P} \to \mathbb{2})$ corresponds to adding an extensional identity type with rules
  - $x\,y : O\,,\ f\,g : A(x,y)\ \vdash\ F(f,g)$ type
  - $x\,y : O\,,\ f : A(x,y)\ \vdash\ s : F(f,f)$
  - $x\,y : O\,,\ f\,g : A(x,y)\,,\ p : F(f,g)\ \vdash\ f = g$
  - $x\,y : O\,,\ f\,g : A(x,y)\,,\ p\,q : F(f,g)\ \vdash\ p = q$

  to the dependent type $x\,y : O\ \vdash\ A(x,y)$ of arrows.

Models in higher types

Models of $\mathcal{T}_1$ in $\mathcal{S}$ are **Segal spaces**, and adding extensional identity types to $\vdash\ O$ or to $x\,y : O\ \vdash\ A(x,y)$ forces the respective types to be 0-truncated. Thus:

$$\infty\text{-Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-Mod}(\mathcal{T}_4) = \{\text{strict 1-categories}\}$$

Part III : The Shape of Contexts

Acknowledgements

This work is inspired by a talk by Per Martin-Löf at the 2014 **Workshop on Constructive mathematics and models of type theory**[14] at the **Institut Henri Poincaré** in Paris.

Thanks for discussions to Mathieu Anel, Carlo Angiuli, Simon Henry, Chaitanya Leena Subramaniam, and Andrew Swan.

[14] https://ihp2014.pps.univ-paris-diderot.fr/doku.php?id=workshop_2

Motivation : Contexts in simple and dependent type theory

Contexts in **simple type theory** are flat:

$x_1 : A_1, \dots, x : nA_n \vdash t(x_1, \dots, x_n) : B$

Contexts in **dependent type theory** are linearly ordered by dependency

$x_1{:}A_1, x_2{:}A_2(x_1), \dots, x_n{:}A(x_1, \dots, x_{n-1}) \vdash B(\vec{x})$

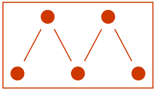
. . . but are they really?

Contexts in the GAT of categories

Consider again the GAT $\mathbb{T}_{\mathsf{Cat}}$ of categories:

$$
\begin{aligned}
&\vdash O \\
\boxed{x\ y : O}\ &\vdash A(x,y) \\
x : O\ &\vdash \mathrm{id}(x) : A(x,x) \\
\boxed{x\,y\,z{:}O\ \ , f{:}A(x,y)\ \ , g{:}A(y,z)}\ &\vdash g\circ f : A(x,z) \\
x\,y : O\,,\ f : A(x,y)\ &\vdash \mathrm{id}(y)\circ f = f \\
x\,y : O\,,\ f : A(x,y)\ &\vdash f\circ \mathrm{id}(x) = f \\
w\,x\,y\,z : O\,,\ e : A(w,x)\,, & \\
f : A(x,y)\,,\ g : A(y,z)\ &\vdash (g\circ f)\circ e = g\circ(f\circ e)
\end{aligned}
$$

The context of $A(x,y)$ has the shape 

The context of composition $g \circ f$ has the shape 

So maybe finite posets are a more realistic representation of dependent contexts than linear orders?

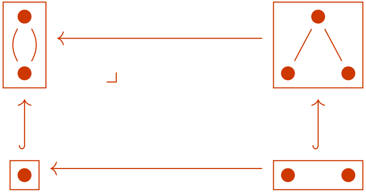
— It turns out that posets are not enough!

The need for non-posetal shapes

Consider the following **pullback** square in the syntactic category $\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]$:

$$\begin{array}{ccc}
(x : O\,,\, f : A(x,x)) & \longrightarrow & (x\,y : O\,,\, f : A(x,y)) \\
\downarrow \lrcorner & & \downarrow \\
(x : O) & \longrightarrow & (x\,y : O)
\end{array}$$

This pullback lives contravariantly over the following **pushout** of shapes:



Taking the pushout in **posets** doesn't give a well-behaved theory, we have to take it in **categories**.

More precisely in the following category of **finite direct categories**.

Finite direct categories

Definition

1. A category $\mathbb{C}$ is called **direct** if there are no infinite inverse paths $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \dots$ of non-identity arrows.
2. A category is called **one-way**, if the only endomorphisms are identities.

Lemma

1. *Direct categories are one-way and skeletal.*
2. *A finite category is direct iff it is one-way and skeletal.*

Definition

FDC is the category of **finite direct categories** and **discrete fibrations**.

The main source is Makkai[15], who writes

> *One-way categories were isolated by F. W. Lawvere[16] [...] Lawvere observed that one-way categories are intimately related to the sketch-based syntax of[17].*

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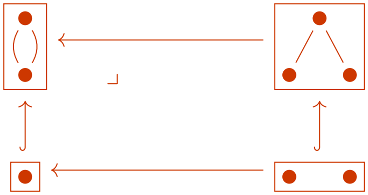
[15] M. Makkai (1995). "First order logic with dependent sorts, with applications to category theory". *Preprint*.
[16] F. William Lawvere (1991). "More on graphic toposes". *Cahiers de Topologie et Géométrie Différentielle Catégoriques*.
[17] M. Makkai (1997). "Generalized sketches as a framework for completeness theorems. I-III". *Journal of Pure and Applied Algebra*.

FDC *as a coclan*

- In this talk we won't pursue the feasability of doing type theory with FDC-shaped contexts (there are obvious issues like definitional equality). Instead we focus on studying the category FDC.
- Among the discrete fibrations, the **injective** ones (a.k.a. **sieve inclusions**) are of special importance: they correspond contravariantly to **context extensions**.
- Sieve inclusions are closed under composition and pullback (along arbitrary maps) in FDC, and the the initial inclusions $\varnothing \hookrightarrow D$ are obviously sieves.



- This means that FDC is a **coclan** (dual to a **clan**) with sieve inclusions as **codisplay maps**.

Models of FDC

Definition

Let $\mathsf{FDC}_0 \subseteq \mathsf{FDC}$ be the full subcategory on FDCs with terminal object ('shapes of types').

Given a model $C : \mathsf{FDC}^{\mathrm{op}} \to \mathsf{Set}$, let C_0 be its restriction to FDC_0. We get the following pullback of categories, where the vertical maps are discrete fibrations:

Claim

The operation $C \mapsto C_0$ gives an equivalence $\mathsf{Mod}(\mathsf{FDC}^{\mathrm{op}}) \simeq [\mathsf{FDC}_0^{\mathrm{op}}, \mathsf{Set}]$.

Locally finite direct categories

Definition

A **locally finite direct category** is a small category $\mathbb{C}$ all of whose slices $\mathbb{C}/c$ are equivalent to finite direct categories.

LFDC is the category of **locally finite direct categories** and **discrete fibrations**.

- For every $C : \mathsf{FDC}_0^{\mathrm{op}} \to \mathsf{Set}$, the category of elements $\mathsf{elts}(C)$ is a LFDC
- Conversely, for every LFDC $\mathbb{C}$, we can define a functor

$$\mathbb{C} \to \mathsf{FDC}_0, \qquad c \mapsto \mathbb{C}/c$$

  all of whose slices are equivalences, meaning that it is a **(Street) fibration of groupoids**.
- We conclude that models of FDC correspond precisely to LFDCs where the fibers of $\mathbb{C}/-$ are (equivalent to) sets, which means that the isomorphisms act freely on incoming non-iso arrows[18].
- Let's call such LFDCs **moderate** – non-moderate LFDCs include groups.

[18] Thanks to Simon Henry for pointing out that the Set-models of FDC do not comprise all LFDCs.

LFDCs vs DLFCs

- **Direct locally finite categories** (DLFCs) are the 0-extensions in LFDC.
- Examples of LFDCs that are not direct:
  - The **index category of symmetric graphs** $0 \rightrightarrows 1 \circlearrowleft$ (with an involution on 1) is locally direct but not direct.
  - FDC_0 is locally finite direct – since $\mathsf{FDC}_0/\mathbb{C} \simeq \mathbb{C}$ for all $\mathbb{C} \in \mathsf{FDC}_0$ – but not direct, since there are automorphisms.
- Construction: For every direct locally finite $\mathbb{D}$, the image of of $\mathbb{D}/- : \mathbb{D} \to \mathsf{FDC}_0$ as a full subcategory is a LFDC which is generally not direct.

$$\mathbb{D} \twoheadrightarrow \mathsf{im}(\mathbb{D}/-) \hookrightarrow \mathsf{FDC}_0$$

For example:

  - For the semisimplex category we obtain FinInj_* – the category of non-empty finite sets and injections
  - for the 'parallel pair category' $\mathbb{P}$, we get the index category of symmetric graphs.

Topos theoretic considerations

- The category $\mathsf{DLFC} \subseteq \mathsf{LFDC}$ is almost a topos: all of its slices are toposes, but it does not have a terminal object!
- To get a topos, we have to go to LFDC, adding automorphisms.
- Analogy: The category Man_{et} of not-necessarily-Hausdorff eucliden Manifolds and local homeomorphisms has topos-slices, but no terminal object. To get a terminal object we have to switch to **locally euclidean toposes**, again introducing automorphisms.
- Adopting Anel/Joyal's topos–logos distinctions, the category of locally euclidean toposes and étale maps is the category of sheaves on the terminal locally euclidean toposes $\mathcal{M}\mathsf{an}$, which can be represented by a site consisting of spaces $\mathbb{R}^n$ and étale maps, where covers are jointly surjective families.
- The topos $\mathcal{M}an$ is **étale-supterminal**, the sense that the forgetful functor

$$\mathsf{Topos}_{et}/\mathcal{M}\mathsf{an} \to \mathsf{Topos}_{\mathsf{et}}$$

  is fully faithful. $[\mathsf{FDC}_0^{\mathsf{op}}, \mathsf{Set}]$ has the same property.
- It seems to be crucial that the full sub-2-categories of $\mathsf{Topos}_{\mathsf{et}}$ on (toposes associated to) FDCs as well as euclidean manifolds are *locally discrete*, since we want to make them into sites.

Thank you for your attention!