Toward the Effective 2-Topos

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Motivation

- In the topos $\mathcal{E}ff$ all maps $\mathbb{N} \to \mathbb{N}$ are computable.
- $\mathcal{E}ff$ contains a small, full subcat $\mathrm{Mod} \subseteq \mathcal{E}ff$ that's
  - internally complete,
  - not a poset.
- The topos $\mathcal{E}ff$ is not Grothendieck.
- A higher version $\mathcal{E}ff^{\infty}$ would be an example of a (non-Grothendieck) elementary ∞-topos.
- But taking e.g. $\mathcal{E}ff^{\infty} = (\mathcal{E}ff^{\Delta^{op}}, \mathrm{Kan})$ does not work...

The Effective 1-Topos

Recall the topos $\mathcal{E}ff$ as an exact completion:

$$\mathrm{PAsm} \rightsquigarrow \mathrm{Asm} \rightsquigarrow \mathcal{E}ff$$

Lex　　　　Regular　　　　Exact

$$\mathrm{Asm} = {}^{\mathrm{reg}}/_{\mathrm{lex}}(\mathrm{PAsm}) \quad \text{free completion}$$

$$\mathcal{E}ff = {}^{\mathrm{ex}}/_{\mathrm{reg}}(\mathrm{Asm}) \quad \text{"} \quad ,$$

$$= {}^{\mathrm{ex}}/_{\mathrm{lex}}(\mathrm{PAsm}) \quad \text{"} \quad .$$

free colim

$\hat{\mathbb{P}}$

y

$\mathrm{PAsm} = \mathbb{P}$

$$
\begin{tikzcd}
\hat{\mathbb{P}} & \\
& \mathrm{Asm} \arrow[ul, hook] \\
\mathbb{P} \arrow[uu, hook, "y"] \arrow[ur, hook] &
\end{tikzcd}
$$

free kernel quotients

$$
\begin{tikzcd}
K \arrow[r, shift left] \arrow[r, shift right] & P \arrow[rr] \arrow[dr, two heads] & & Q \\
& & P/K \arrow[ur] &
\end{tikzcd}
$$

$$\begin{tikzcd}
\hat{\mathbb{P}} & \\
& \mathrm{Eff} \arrow[ul, hook] \\
& \mathrm{Asm} \arrow[u, hook] \\
\mathbb{P} \arrow[uuu, hook, "y"] \arrow[ur, hook]
\end{tikzcd}$$

free exact quotients

$$E \rightrightarrows P \twoheadrightarrow P/E$$

free kernel quotients

$$\begin{tikzcd}
K \arrow[r, shift left] \arrow[r, shift right] & P \arrow[rr] \arrow[dr, two heads] & & Q \\
& & P/K \arrow[ur, tail] &
\end{tikzcd}$$

Factorization of Yoneda

$$
\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, hook] & & & & \hat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \mathcal{E}ff \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}) \arrow[u]
\end{tikzcd}
$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, hook] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \cancel{\mathcal{E}ff}\ \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}) \arrow[u]
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\end{tikzcd}
$$

indecomposable projectives:

- $I = X + Y \Rightarrow I = X$ or $I = Y$
- $$
\begin{tikzcd}
 & E \arrow[d, two heads] \\
P \arrow[ur, dashed] \arrow[r] & X
\end{tikzcd}
$$

$\Longleftrightarrow$ "representable" $P = yp$ f. some $p \in \mathbb{P}$.

Factorization of Yoneda

$$
\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\cong"] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm.}) \arrow[u]
\end{tikzcd}
$$

P, Q $\qquad\qquad$ $P \twoheadrightarrow K$

K (super)compact:

$$
\left.
\begin{array}{l}
(X_i \longrightarrow K)_i \text{ covers} \\
\Rightarrow \; X_k \twoheadrightarrow K \text{ f. some } k
\end{array}
\right\}
\iff
\begin{array}{l}
P \twoheadrightarrow K \\
\text{f. some } P
\end{array}
$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\wr"] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm.}) \arrow[u]
\end{tikzcd}$$

P, Q $\qquad P \twoheadrightarrow A \rightarrowtail Q$ $\qquad P \twoheadrightarrow K$

Assemblies:

$$\left.\begin{array}{c} P \twoheadrightarrow A \rightarrowtail Q \\ \text{for ind.proj. } P, Q \end{array}\right\} \iff \begin{array}{l} A \text{ cpt } \& \\ A \rightarrowtail Q \end{array}$$

Factorization of Yoneda

$$
\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\cong" description] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm.}) \arrow[u, hook]
\end{tikzcd}
$$

$P, Q \qquad P \twoheadrightarrow A \rightarrowtail P' \qquad A' \rightrightarrows A \twoheadrightarrow C \qquad P \twoheadrightarrow K$

C <u>coherent</u> :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\wr"] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}.) \arrow[u]
\end{tikzcd}$$

P, Q $\qquad$ $P \twoheadrightarrow A \rightarrowtail P'$ $\qquad$ $A' \rightrightarrows A \twoheadrightarrow C$ $\qquad$ $P \twoheadrightarrow K$

C <u>coherent</u> :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

where

$f : X \to Y$ is <u>cpt</u> :

$$\begin{tikzcd}
K' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
K \arrow[r] & Y
\end{tikzcd}$$

Factorization of Yoneda

$$
\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, sloped, "\cong"] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \text{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \text{Sh(Asm.)} \arrow[u, hook]
\end{tikzcd}
$$

$$P, Q \qquad P \twoheadrightarrow A \rightarrowtail P' \qquad A' \rightrightarrows A \twoheadrightarrow C \qquad P \twoheadrightarrow K$$

C coherent :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

$$\left.\begin{array}{l} \bullet\ C \text{ is cpt.} \\ \bullet\ C \xrightarrow{\Delta} C\times C \text{ cpt.} \end{array}\right\} \iff A' \rightrightarrows A \twoheadrightarrow C \text{ exact}.$$

Factorization of Yoneda

$$
\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\cong" rotate=90] & & & & \widehat{\mathbb{P}} \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm}) \arrow[u]
\end{tikzcd}
$$

P, Q $\qquad$ $P \twoheadrightarrow A \rightarrowtail P'$ $\qquad$ $A' \rightrightarrows A \twoheadrightarrow C$ $\qquad$ $P \twoheadrightarrow K$

F sheaf on Asm (for the reg. epi. top.) $\iff$

$$
\begin{tikzcd}
A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r] \arrow[d, two heads] & F \\
& B \arrow[ur, dashed] &
\end{tikzcd}
$$

Factorization of Yoneda

$$\begin{tikzcd}
\mathbb{P} \arrow[rrrr, hook, "y"] \arrow[d, phantom, "\cong"] & & & & \widehat{\mathbb{P}} \arrow[d, phantom, "\cong"] \\
\text{ind. proj.} \arrow[r, hook] & \mathrm{Asm} \arrow[r, hook] & \text{coh.} \arrow[r, hook] & \text{cpt.} \arrow[r, hook] & \mathrm{Sh}(\mathrm{Asm})
\end{tikzcd}$$

$P, Q \qquad P \twoheadrightarrow A \rightarrowtail P' \qquad A' \rightrightarrows A \twoheadrightarrow C \qquad P \twoheadrightarrow K$

F sheaf on Asm (for the reg. epi. top.) $\iff$

$$\begin{tikzcd}
A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r] \arrow[d, two heads] & F \\
& B \arrow[ur, dashed] &
\end{tikzcd}$$

$\iff \quad F \in \widehat{\mathbb{P}} = [\mathbb{P}^{op}, \mathrm{Set}]$.

Thm (Lack) For a reg. Cat. $\mathcal{R}$,

$$\mathcal{R}_{ex/reg} \subseteq Sh(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \rightrightarrows yA \twoheadrightarrow E \text{ ex.}$$

$$\text{f. } A', A \in \mathcal{R} \rangle.$$

Thm (Lack) For a reg. cat. $\mathcal{R}$,

$$\mathcal{R}_{ex/reg} \subseteq Sh(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid yA' \rightrightarrows yA \twoheadrightarrow E \text{ ex.} \rangle$$

f. $A', A \in \mathcal{R}$.

Cor. $Sh(Asm)_{coh} = ex/reg(Asm)$.

Thm (Lack) For a reg. cat. $\mathcal{R}$,

$$\mathcal{R}_{ex/reg} \subseteq Sh(\mathcal{R}, \text{reg epi})$$
$$= \langle E \mid yA' \rightrightarrows yA \twoheadrightarrow E \text{ ex.} \rangle$$
$$\text{f. } A, A' \in \mathcal{R}$$

Cor. $Sh(Asm)_{coh} = ex/reg(Asm) = \mathcal{E}ff$!

Summary

$$
\begin{array}{ccccccc}
\mathrm{PAsm} & \subset & \mathrm{Asm} & \subset & \mathcal{E}ff & \subset & \widehat{\mathrm{PAsm}} \\
 & & & & \wr\!\wr & & \wr\!\wr \\
 & & & & \mathrm{CohSh}(\mathrm{Asm}) & \subset & \mathrm{Sh}(\mathrm{Asm})
\end{array}
$$

Summary

$$\mathrm{PAsm} \subset \mathrm{Asm} \subset \mathcal{E}ff \subset \widehat{\mathrm{PAsm}}$$

Now in order to get a 2-topos we'll take <u>internal groupoids</u> in the Groth. topos $\widehat{\mathrm{PAsm}}$.

Summary

$$\mathrm{PAsm} \subset \mathrm{Asm} \subset \mathcal{E}ff \subset \widehat{\mathrm{PAsm}}$$

Now in order to get a 2-topos we'll take <u>internal groupoids</u> in the Groth. topos $\widehat{\mathrm{PAsm}}$,

$$\mathrm{Gpd}(\widehat{\mathbb{P}}) = [\mathbb{P}^{op}, \mathrm{Gpd}] .$$

Finally, we shall restrict $\mathrm{Gpd}(\widehat{\mathbb{P}})$ to a subcat of "coherent groupoids" such that

$$\begin{tikzcd}
\mathrm{CohGpd} \arrow[r, hook] & \mathrm{Gpd} \\
\mathrm{Eff} \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\urcorner" very near start] & \mathrm{Gpd}_0 \arrow[u, hook]
\end{tikzcd}$$

in $\widehat{\mathbb{P}}$.

QMS on $Gpd(\mathcal{E})$

There are different QMS on $Gpd(\mathcal{E})$ for Groth. topos $\mathcal{E}$. We use one that:

(1) admits a model of HoTT,

(2) has $\mathcal{E}ff$ as the coherent 0-types:

$$\mathcal{E}ff = CohGpd(\mathcal{E})_0 \subseteq Gpd(\mathcal{E})_0 \subseteq Gpd(\mathcal{E}).$$

For (1), use Shulman's thm. on ∞-toposes:

Thm. There's a type theoretic model stucture on $Gpd(\hat{\mathbb{P}})$ with:

- Weak equivalences = objectwise equivalences of cats,
- Cofibrations = objectwise injections on objects,
- Fibrations = algebraic "cobar" fibrations.

Note: Quillen equiv. to the J-T strong stacks QMS.

Coherent Groupoid

For (2):

Def. A groupoid $\mathbb{G} = (G_1 \rightrightarrows G_0)$ in $\widehat{\mathbb{P}}$ is <u>coherent</u> if:

- $\mathbb{G}$ is cpt,
- $\Delta_1 : \mathbb{G} \longrightarrow \mathbb{G} \times \mathbb{G}$ is cpt,
- $\Delta_2 : \Delta_1 \longrightarrow \Delta_1 \times \Delta_1$ is cpt.

Coherent Groupoid

For (2):

Def. A groupoid $\mathbb{G} = (G_1 \rightrightarrows G_0)$ in $\widehat{\mathbb{P}}$ is <u>coherent</u> if:

- $\mathbb{G}$ is h-cpt, $P \to \mathbb{G}$ eso
- $\Delta_1 : \mathbb{G} \longrightarrow \mathbb{G} \times \mathbb{G}$ is h-cpt,
- $\Delta_2 : \Delta_1 \longrightarrow \Delta_1 \times \Delta_1$ is h-cpt.

Prop. Let $\mathbb{G}$ be a coherent gpd and a 0-type $\Delta_1 : \mathbb{G} \rightarrowtail \mathbb{G} \times \mathbb{G}$. Then

- $\pi_0 \mathbb{G}$ is in $\mathcal{E}\mathit{ff}_s$
- $\mathbb{G} \underset{\text{w.e.}}{\simeq} \pi_0 \mathbb{G}$.

Prop. If $\mathbb{G}$ is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since $\mathbb{G}$ is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P', since Δ cpt.

$$\begin{tikzcd}
P' \arrow[r, twoheadrightarrow] \arrow[dr] & \mathbb{K} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{G}^I \arrow[d, "\tilde{\Delta}"] \\
 & P \times P \arrow[r, twoheadrightarrow] & \mathbb{G} \times \mathbb{G}
\end{tikzcd}$$

<u>Prop.</u> If $\mathbb{G}$ is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since $\mathbb{G}$ is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P', since Δ cpt.

Since $\mathbb{G}$ is a 0-type,

$\Delta : \mathbb{G} \longrightarrow \mathbb{G} \times \mathbb{G}$ is monic.

$$
\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & \mathbb{K} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^I \arrow[d, tail, "\tilde{\Delta}"] \\
 & P \times P \arrow[r] & \mathbb{G} \times \mathbb{G}
\end{tikzcd}
$$

<u>Prop.</u> If $\mathbb{G}$ is a coherent 0-type, then $\Pi_0\mathbb{G}$ is coherent.

Pf. Since $\mathbb{G}$ is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P', since Δ cpt.

Since $\mathbb{G}$ is a 0-type,

$\tilde{\Delta} : \mathbb{G}^I \longrightarrow \mathbb{G} \times \mathbb{G}$ is monic.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr, two heads] & \mathbb{K} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^I \arrow[d, tail, "\tilde{\Delta}"] \\
 & P \times P \arrow[r, two heads] & \mathbb{G} \times \mathbb{G}
\end{tikzcd}$$

Hence so is $\mathbb{K} \rightarrowtail P \times P$,

so $\mathbb{K}$ is an assembly.

Prop. If $\mathbb{G}$ is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since $\mathbb{G}$ is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P'.

Since $\mathbb{G}$ is a 0-type $\Delta : \mathbb{G} \to \mathbb{G} \times \mathbb{G}$ is monic.

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so $\mathbb{K}$ is an assembly.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr, two heads] & \mathbb{K} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^I \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathbb{G}} \arrow[d, tail] \\
 & P \times P \arrow[r, two heads] & \mathbb{G} \times \mathbb{G} \arrow[r, two heads] & \overline{\mathbb{G}} \times \overline{\mathbb{G}}
\end{tikzcd}$$

Write $\overline{\mathbb{G}} = \pi_0 \mathbb{G} = G_0 / G_1$.

<u>Prop.</u> If $\mathbb{G}$ is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since G_0 is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P'.

Since $\mathbb{G}$ is a 0-type

$\mathbb{G} \to \mathbb{G} \times \mathbb{G}$ is monic.

Hence so is $K \rightarrowtail P \times P$,

so K is an assembly.

Write $\overline{\mathbb{G}} = \pi_0 \mathbb{G} = G_0 / G_1$.

$$\begin{tikzcd}
P' \arrow[r, twoheadrightarrow] \arrow[dr] & K \arrow[r] \arrow[d, rightarrowtail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^I \arrow[r] \arrow[d, rightarrowtail] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathbb{G}} \arrow[d, rightarrowtail] \\
 & P \times P \arrow[r, twoheadrightarrow] & \mathbb{G} \times \mathbb{G} \arrow[r, twoheadrightarrow] & \overline{\mathbb{G}} \times \overline{\mathbb{G}}
\end{tikzcd}$$

We then have $K \rightrightarrows P \twoheadrightarrow \overline{\mathbb{G}}$ exact.

Prop. If $\mathbb{G}$ is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since $\mathbb{G}$ is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P'.

Since $\mathbb{G}$ is a 0-type $\mathbb{G} \to \mathbb{G} \times \mathbb{G}$ is monic.

Hence so is $K \rightarrowtail P \times P$,

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Write $\overline{\mathbb{G}} = \pi_0 \mathbb{G} = G_0 / G_1$.

We then have $K \rightrightarrows P \twoheadrightarrow \overline{\mathbb{G}}$ exact.

$$
\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & K \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^I \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathbb{G}} \arrow[d, tail] \\
& P \times P \arrow[r, two heads] & \mathbb{G} \times \mathbb{G} \arrow[r, two heads] & \overline{\mathbb{G}} \times \overline{\mathbb{G}}
\end{tikzcd}
$$

So $\overline{\mathbb{G}} = \pi_0 \mathbb{G}$ is coherent.

__Prop.__ If $\mathbb{G}$ is a coherent 0-type, then $\pi_0\mathbb{G}$ is coherent.

Pf. Since G_0 is cpt it has a cover $P \twoheadrightarrow \mathbb{G}$.

Take the p.b. $\mathbb{K}$ and its cover P'.

Since $\mathbb{G}$ is a 0-type

$\mathbb{G} \longrightarrow \mathbb{G} \times \mathbb{G}$ is monic.

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so K is an assembly.

$$\begin{tikzcd}
P' \arrow[r, two heads] \arrow[dr] & K \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}^I \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{G} \arrow[d, tail] \\
& P \times P \arrow[r, two heads] & \mathbb{G} \times \mathbb{G} \arrow[r, two heads] & \overline{G} \times \overline{G}
\end{tikzcd}$$

Write $\overline{G} = \pi_0\mathbb{G} = G_0/G_1$.

We then have $K \rightrightarrows P \twoheadrightarrow \overline{G}$ exact.

So $\overline{G} = \pi_0\mathbb{G}$ is coherent.

Finally, $\mathbb{G} \simeq \overline{\mathbb{G}}$ is a weak equivalence. $\square$

Thm Over $\hat{\mathbb{P}}$, we have

$$
\begin{tikzcd}
\mathcal{E}\!f\!f \simeq \mathrm{CohGpd}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Gpd}_0 \arrow[d] \\
\mathrm{CohGpd} \arrow[r] & \mathrm{Gpd}
\end{tikzcd}
.
$$

Thm Over $\hat{\mathbb{P}}$, we have

$$\begin{tikzcd} \mathcal{E}ff \simeq \mathrm{CohGpd}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gpd}_0 \arrow[d] \\ \mathrm{CohGpd} \arrow[r] & \mathrm{Gpd} \end{tikzcd}$$

Take $\mathcal{E}ff^2 := \mathrm{CohGpd} \longrightarrow \mathrm{Gpd}$.

to get an elementary (2,1)-topos with

$$\mathcal{E}ff^2_0 \simeq \mathcal{E}ff.$$

THANKS !