

Toward the Effective λ -Topos

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jww/

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Motivation

- In the topos $\mathcal{E}ff$ all maps $\mathbb{N} \rightarrow \mathbb{N}$ are **computable**.
- $\mathcal{E}ff$ contains a **small, full subcat** $Mod \subseteq \mathcal{E}ff$ that's
 - **internally complete**,
 - **not a poset**.
- The topos $\mathcal{E}ff$ is **not** Grothendieck.
- A higher version $\mathcal{E}ff^\infty$ would be an example of a (non-Grothendieck) **elementary ∞ -topos**.
- **But** taking e.g. $\mathcal{E}ff^\infty = (\mathcal{E}ff^{\Delta^{cp}}, Kan)$ does **not** work...

The Effective 1-Topos

Recall the topos $\mathcal{E}ff$ as an exact completion:

$$P\mathcal{A}sm \rightsquigarrow \mathcal{A}sm \rightsquigarrow \mathcal{E}ff$$

Lex

Regular

Exact

$$\mathcal{A}sm = \text{reg}_{\text{lex}}(P\mathcal{A}sm) \quad \text{free completion}$$

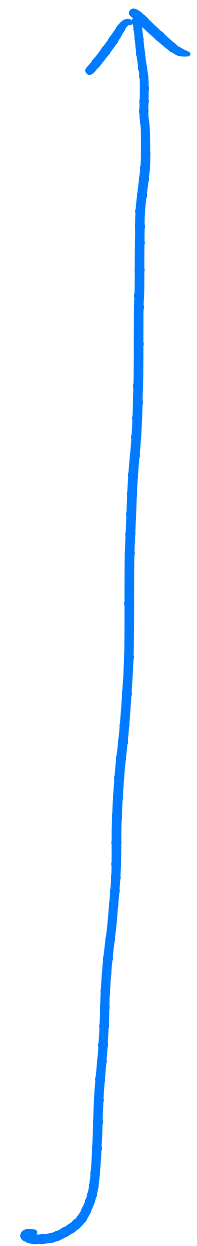
$$\mathcal{E}ff = \text{ex}_{\text{reg}}(\mathcal{A}sm) \quad \text{"}$$

$$= \text{ex}_{\text{lex}}(P\mathcal{A}sm) \quad \text{"}$$

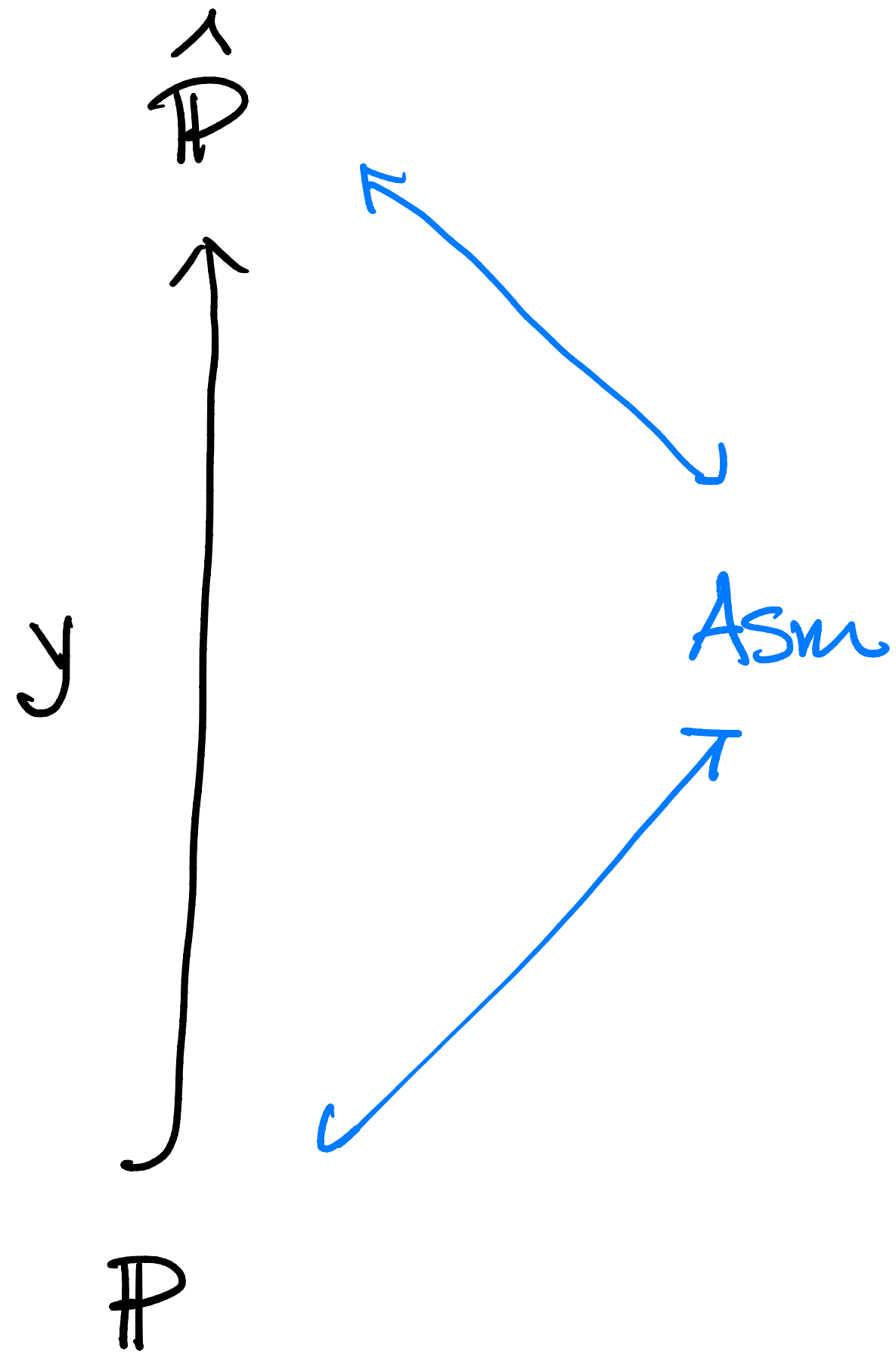
free
colim

\mathbb{P}

\hookrightarrow

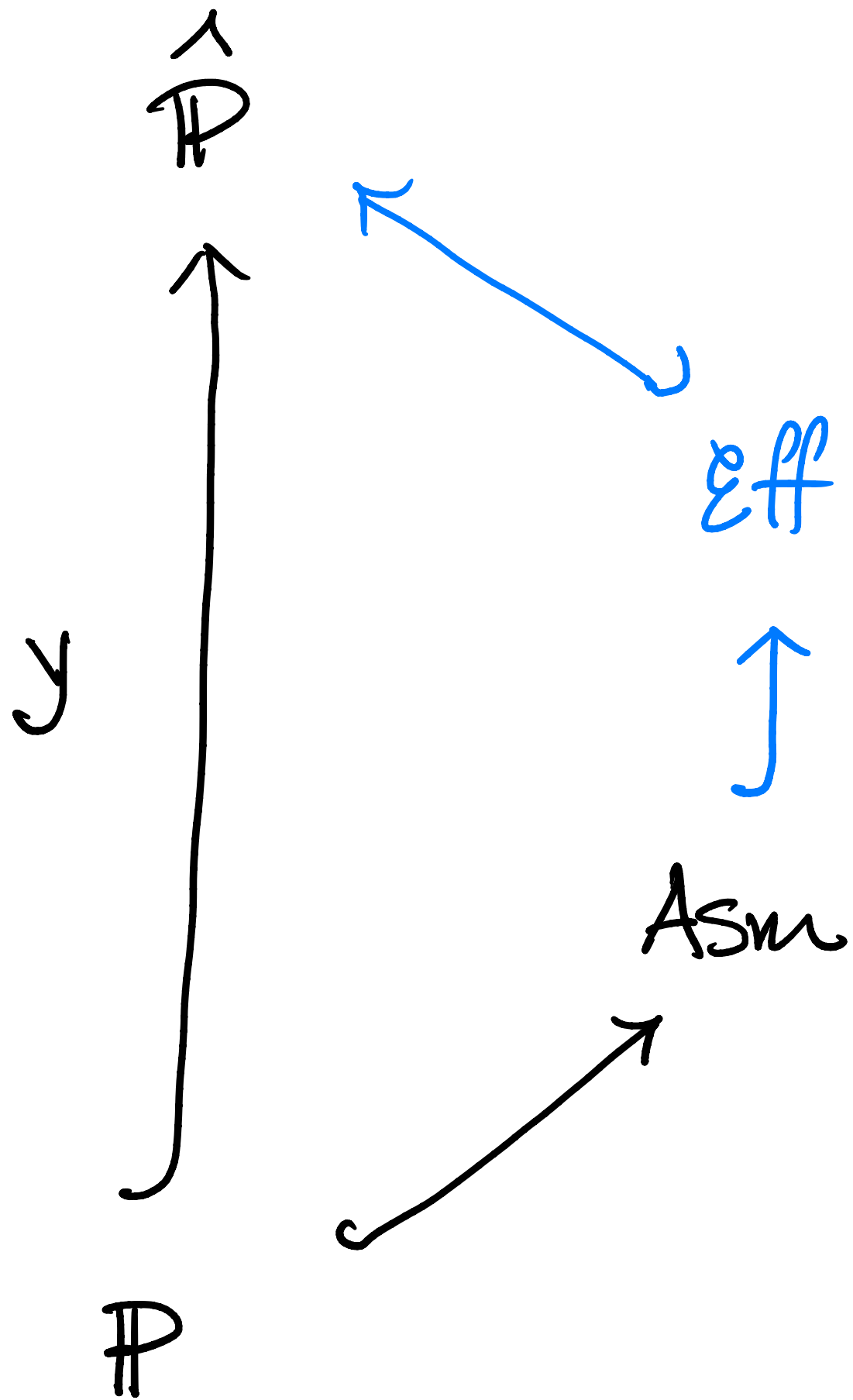


$\text{PAsm} = \mathbb{P}$



free kernel quotients

$$\begin{array}{ccccc}
 K & \xrightarrow{\quad} & P & \xrightarrow{\quad} & Q \\
 & & \searrow & & \nearrow \\
 & & P/K & &
 \end{array}$$



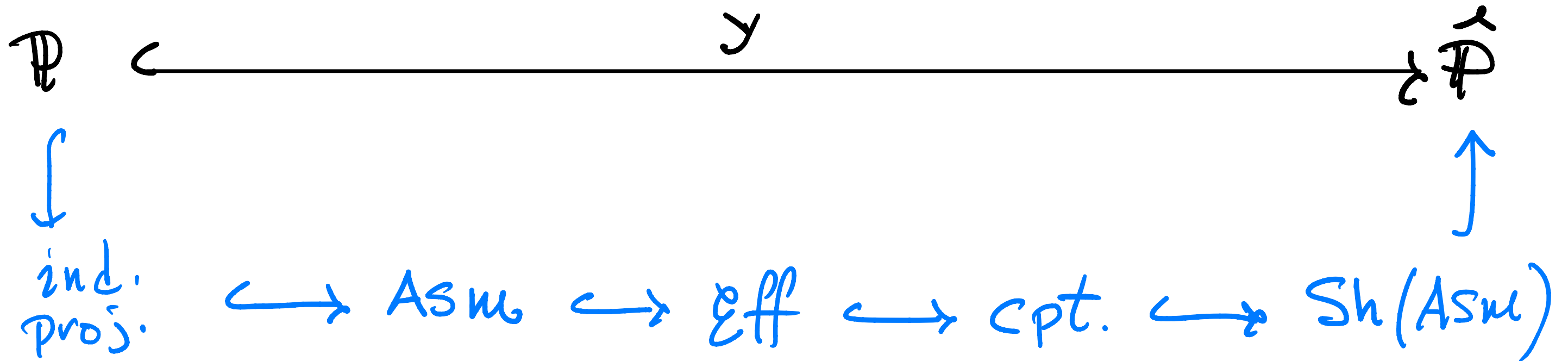
free exact quotients

$$E \rightrightarrows P \twoheadrightarrow P/E$$

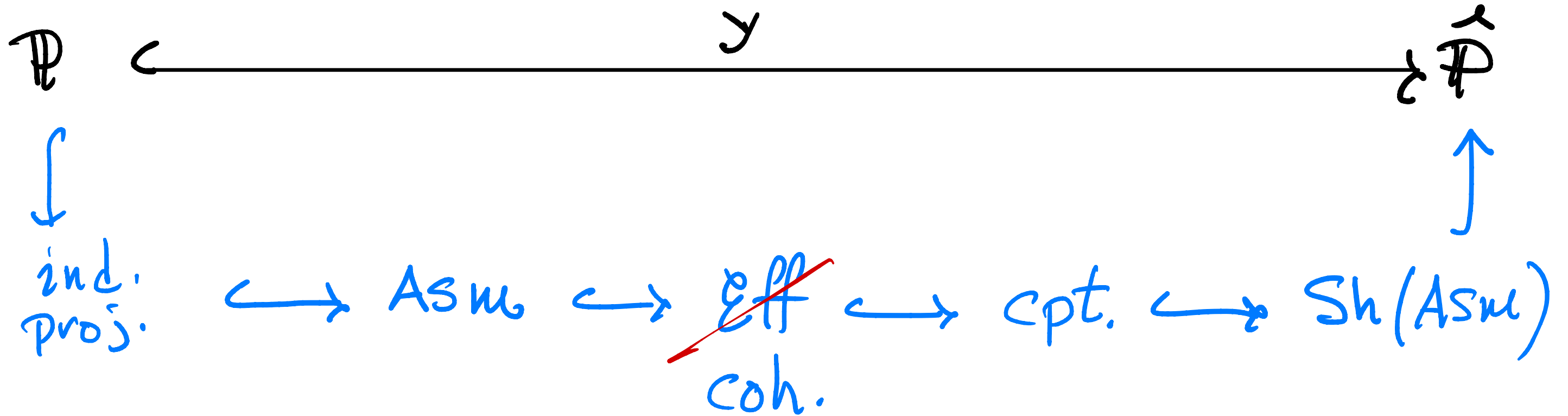
free kernel quotients

$$K \rightrightarrows P \begin{matrix} \longrightarrow Q \\ \searrow \twoheadrightarrow P/K \nearrow \end{matrix}$$

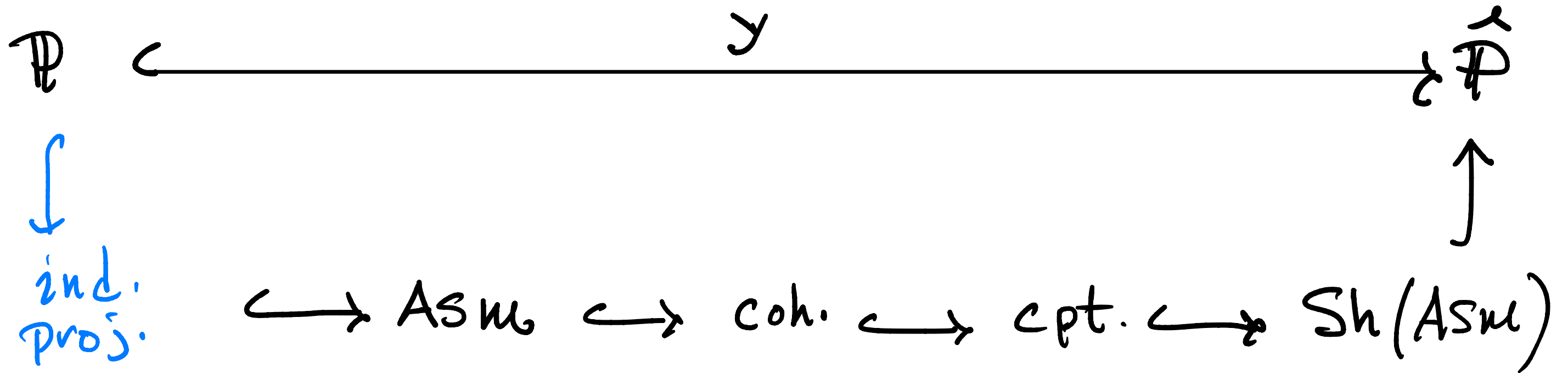
Factorization of Yoneda



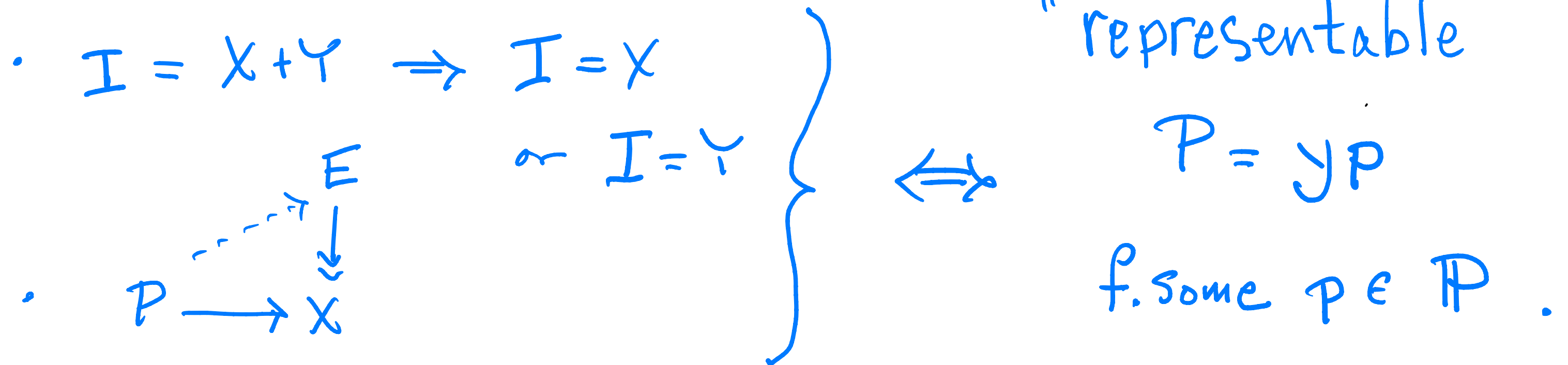
Factorization of Yoneda



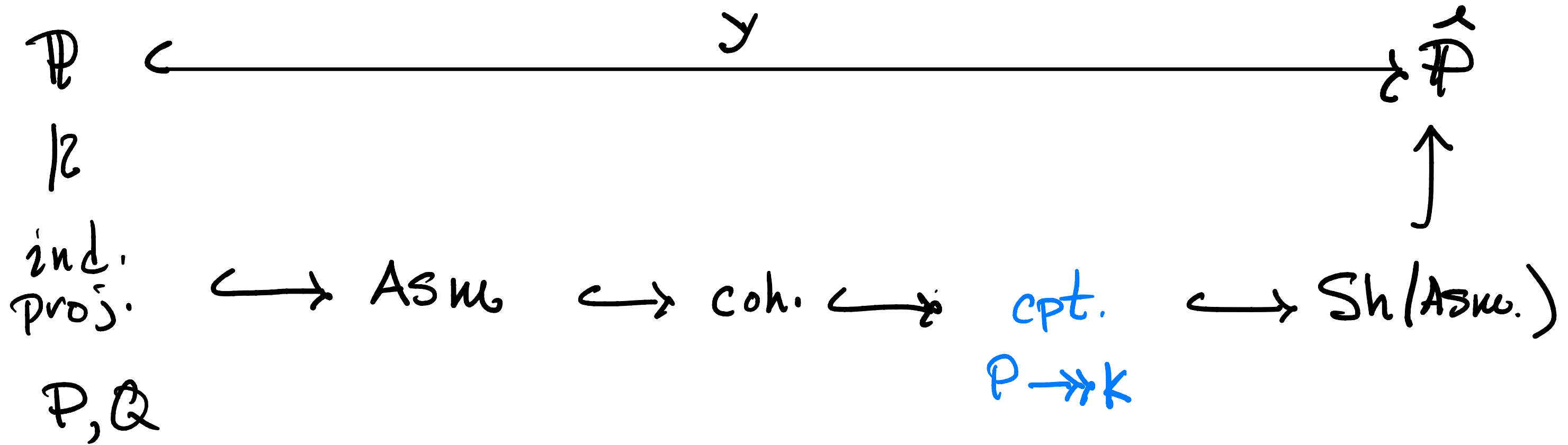
Factorization of Yoneda



indecomposable projectives:



Factorization of Yoneda



K (super)compact:

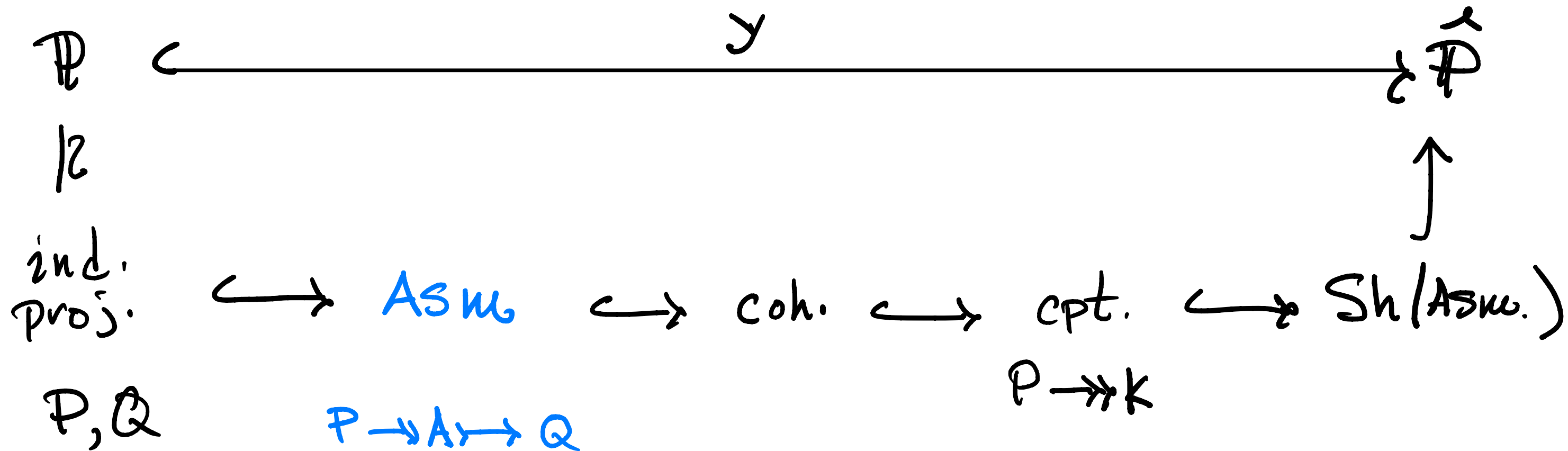
$(X_i \rightarrow K)_i$ covers

$\Rightarrow X_k \twoheadrightarrow K$ f. some k

\Leftrightarrow

$P \twoheadrightarrow K$
f. some P

Factorization of Yoneda



Assemblies:

$$P \twoheadrightarrow A \twoheadrightarrow Q$$

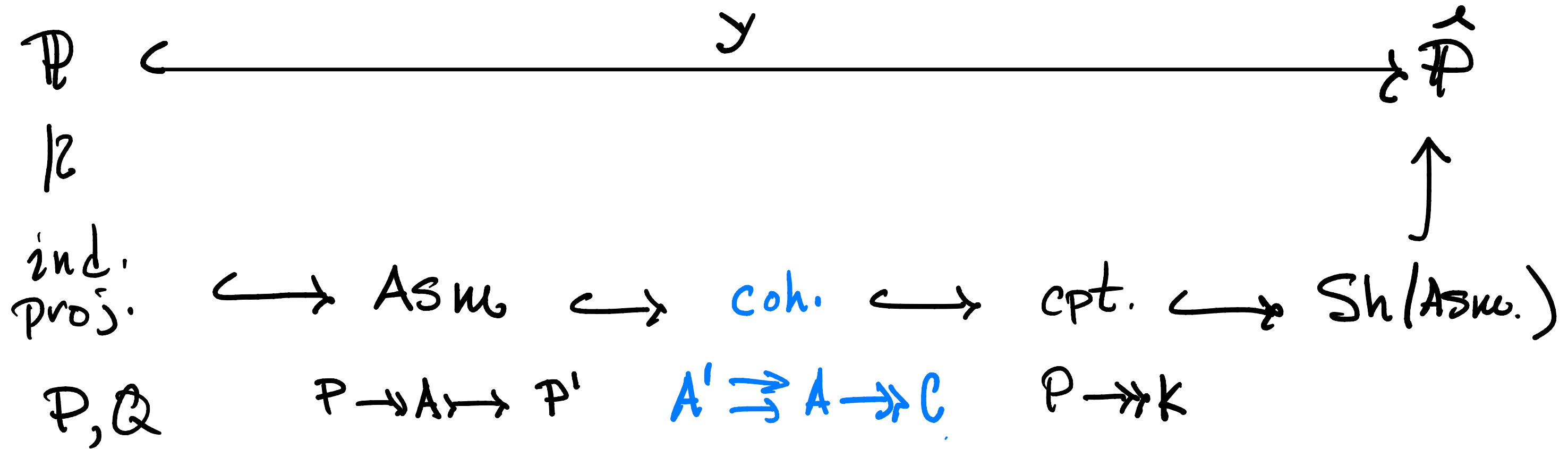
for ind. proj. P, Q

\Leftrightarrow

$$A \text{ cpt } \&$$

$$A \twoheadrightarrow Q$$

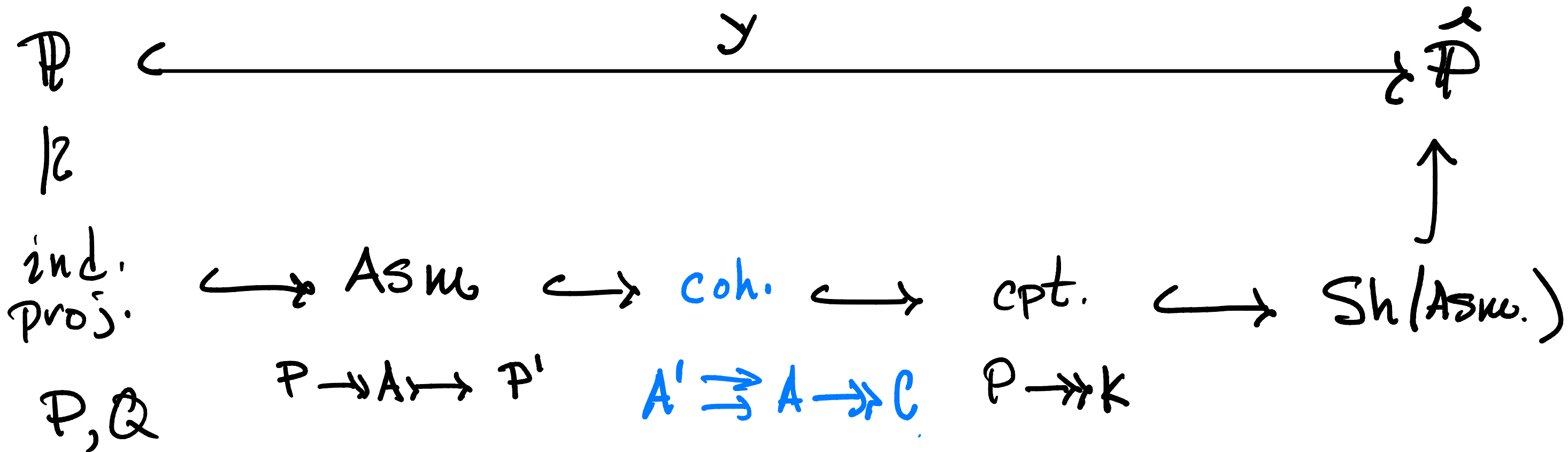
Factorization of Yoneda



C coherent :

- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

Factorization of Yoneda



C Coherent :

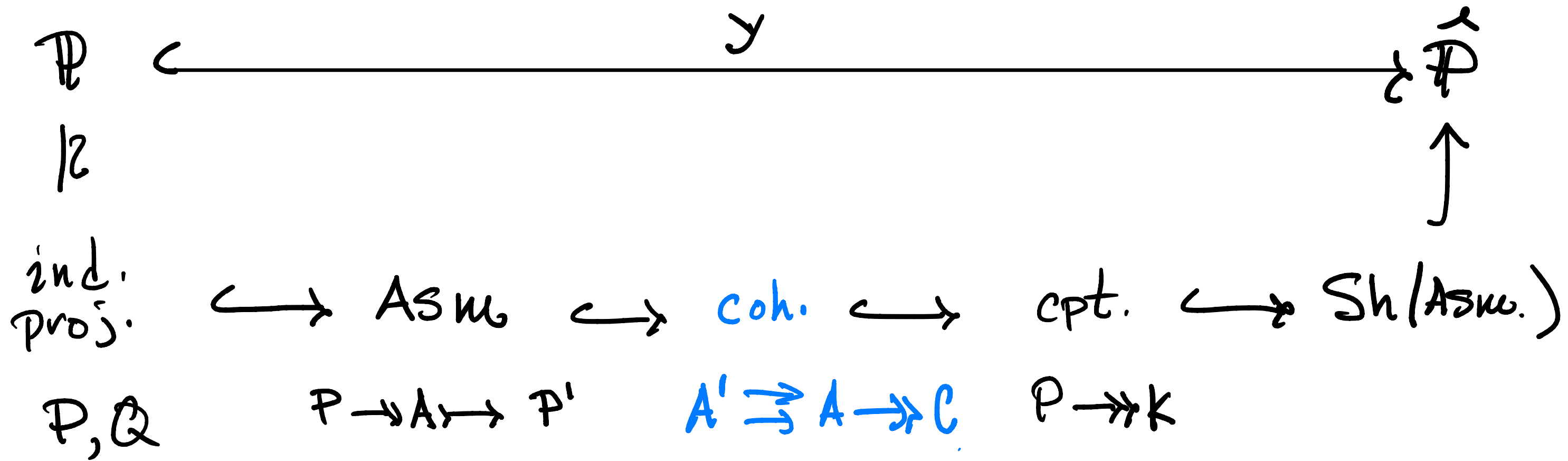
- C is cpt.
- $C \xrightarrow{\Delta} C \times C$ cpt.

where

$f: X \rightarrow \mathcal{Y}$ is cpt :

$$\begin{array}{ccc}
 \mathcal{K}' & \longrightarrow & X \\
 \downarrow & \lrcorner & \downarrow f \\
 \mathcal{K} & \longrightarrow & \mathcal{Y}
 \end{array}$$

Factorization of Yoneda

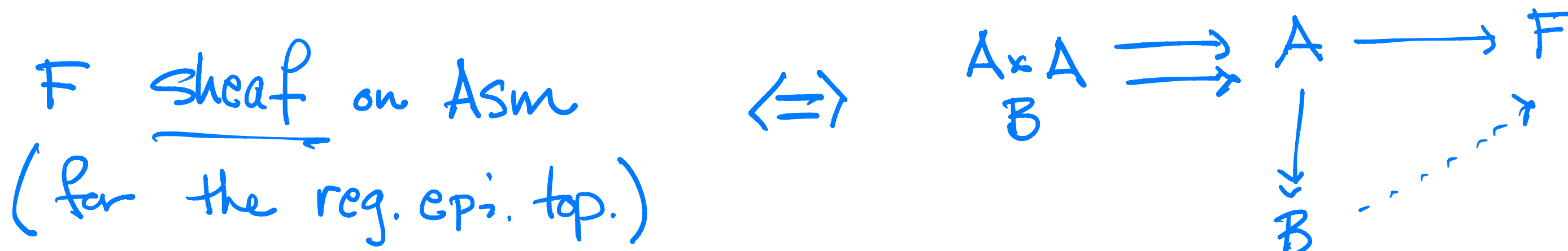
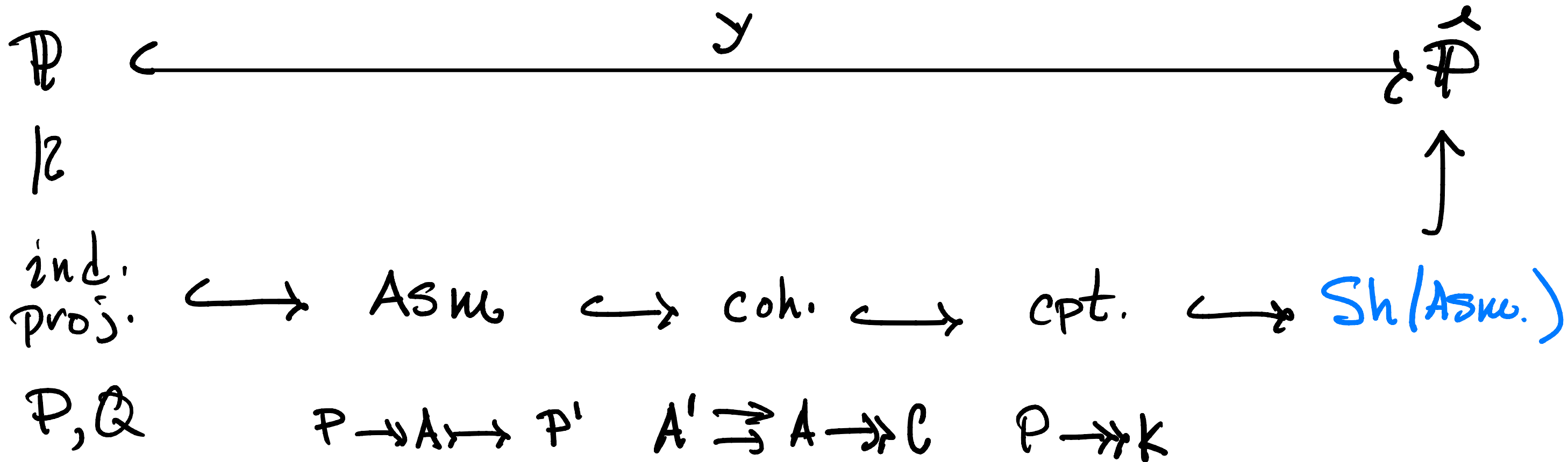


\mathcal{C} Coherent :

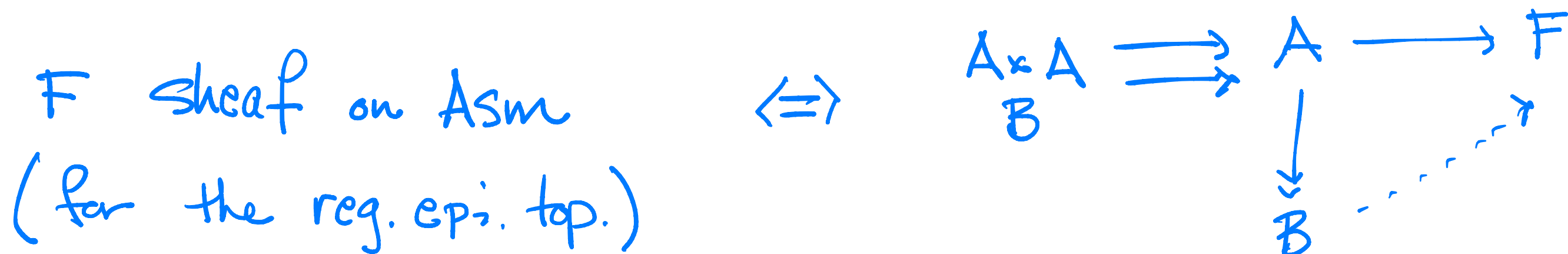
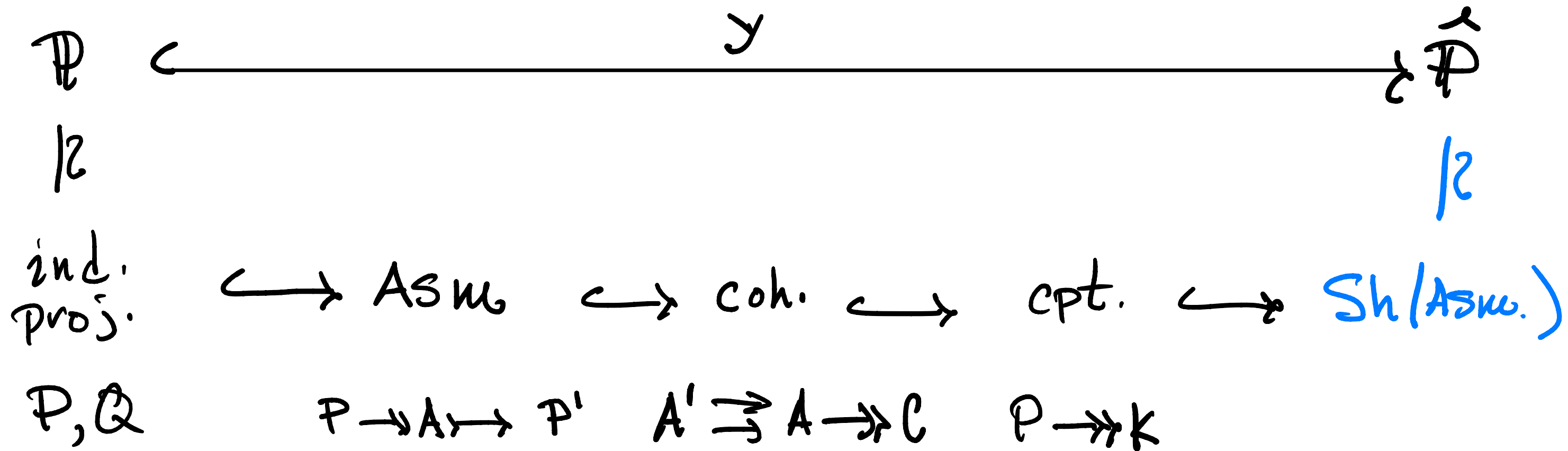
- \mathcal{C} is cpt.
- $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C}$ cpt.

$$\iff \mathcal{A}' \twoheadrightarrow \mathcal{A} \twoheadrightarrow \mathcal{C} \text{ exact}$$

Factorization of Yoneda



Factorization of Yoneda



$\Leftrightarrow F \in \hat{\mathcal{P}} = [\mathcal{P}^{op}, \text{Set}] .$

Thm (Lack)

For a reg. cat. \mathcal{R} ,

$$\mathcal{R}_{\text{ex/reg}} \subseteq \text{Sh}(\mathcal{R}, \text{reg epi})$$

$$= \langle E \mid y_{A'} \rightrightarrows y_A \twoheadrightarrow E \text{ ex.} \rangle$$

f. $A', A \in \mathcal{R}$

Thm (Lack)

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f. $A', A \in \mathcal{R}$

Cor.

$$\text{Sh}(\text{Asm})_{\text{coh}} = \text{ex/reg}(\text{Asm}).$$

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$$= \langle E \mid y_{A'} \rightrightarrows y_A \twoheadrightarrow E \text{ ex.} \rangle$$

f. $A, A' \in \mathcal{R}$

Cor. $\text{Sh}(\text{Asm})_{\text{coh}} = \text{ex/reg}(\text{Asm}) = \text{eff}!$

Summary

$$PAsm \subset Asm \subset \underset{\mathbb{Z}}{Eff} \subset \overset{\wedge}{\underset{\mathbb{Z}}{PAsm}}$$
$$CohSh(Asm) \subset Sh(Asm)$$

Summary

$$\mathbf{PAsm} \subset \mathbf{Asm} \subset \mathbf{Eff} \subset \widehat{\mathbf{PAsm}}$$

Now in order to get a 2-topos we'll take
internal groupoids in the Groth. topos $\widehat{\mathbf{PAsm}}$.

Summary

$$\mathcal{P}Asm \subset Asm \subset Eff \subset \widehat{\mathcal{P}Asm}$$

Now in order to get a 2-topos we'll take
internal groupoids in the Groth. topos $\widehat{\mathcal{P}Asm}$,

$$Gpd(\widehat{\mathcal{P}}) = [\mathcal{P}^{\text{op}}, Gpd] .$$

Finally, we shall restrict $\text{Gpd}(\hat{\mathcal{P}})$ to a subcat of "coherent groupoids" such that

$$\begin{array}{ccc}
 \text{Coh Gpd} & \hookrightarrow & \text{Gpd} \\
 \uparrow & \lrcorner & \uparrow \\
 \text{Eff} & \hookrightarrow & \text{Gpd}_0
 \end{array}
 \quad \text{in } \hat{\mathcal{P}}$$

QMS on $\mathcal{Gpd}(\mathcal{E})$

There are different QMS on $\mathcal{Gpd}(\mathcal{E})$ for Groth. tops \mathcal{E} . We use one that:

- (1) admits a model of $\mathbb{H}T$,
- (2) has $\mathcal{E}ff$ as the coherent σ -types:

$$\mathcal{E}ff = \text{Coh}\mathcal{Gpd}(\mathcal{E})_0 \subseteq \mathcal{Gpd}(\mathcal{E})_0 \subseteq \mathcal{Gpd}(\mathcal{E}).$$

For (1), use Shulman's thm. on ∞ -toposes:

Thm. There's a type theoretic model structure
on $\text{Gpd}(\hat{\mathbb{P}})$ with:

- Weak equivalences = objectwise equivalences of cats,
- Cofibrations = objectwise injections on objects,
- Fibrations = algebraic "cobar" fibrations.

Note: Quillen equiv. to the J-T Strong Stacks QMS.

Coherent Groupoid

For (2):

Def. A groupoid $\mathbb{G} = (G, \rightrightarrows G_0)$ in $\hat{\mathcal{D}}$ is
coherent if :

- \mathbb{G} is cpt ,
- $\Delta_1 : \mathbb{G} \longrightarrow \mathbb{G} \times \mathbb{G}$ is cpt ,
- $\Delta_2 : \Delta_1 \longrightarrow \Delta_1 \times \Delta_1$ is cpt .

Coherent Groupoid

For (2):

Def. A groupoid $\mathbb{G} = (G, \rightrightarrows G_0)$ in $\hat{\mathcal{D}}$ is coherent if:

- \mathbb{G} is h -cpt, $\mathcal{P} \rightarrow \mathbb{G}$ eso
- $\Delta_1: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ is h -cpt,
- $\Delta_2: \Delta_1 \rightarrow \Delta_1 \times \Delta_1$ is h -cpt.

Prop.

Let \mathbb{G} be a coherent gpd and a 0-type $\Delta_1: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$. Then

• $\pi_0 \mathbb{G}$ is in Eff ,

• $\mathbb{G} \underset{\text{w.e.}}{\simeq} \pi_0 \mathbb{G}$.

Prop. If G is a coherent O -type, then $\pi_0 G$ is coherent.

Pf. Since G is cpt it has a cover $P \rightarrow G$.

Take the p.b. \mathbb{K} and its cover P' , since Δ cpt .

$$\begin{array}{ccccc} P' & \rightarrow & \mathbb{K} & \rightarrow & G^I \\ & \searrow & \downarrow & \searrow & \downarrow \tilde{\Delta} \\ & & P \times P & \rightarrow & G \times G \end{array}$$

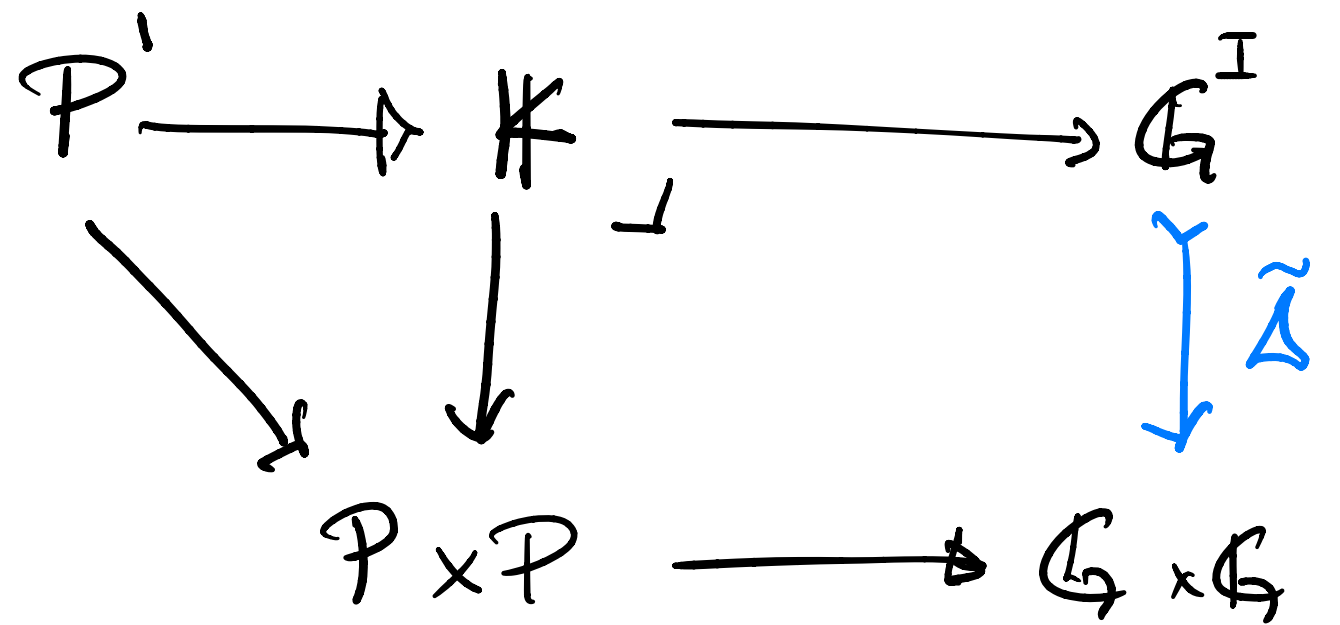
Prop. If \mathbb{G} is a coherent \mathcal{O} -type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since \mathbb{G} is cpt it has a cover $P \rightarrow \mathbb{G}$.

Take the p.b. \mathbb{H} and its cover P' , since Δ cpt.

Since \mathbb{G} is a \mathcal{O} -type,

$\Delta: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ is monic.

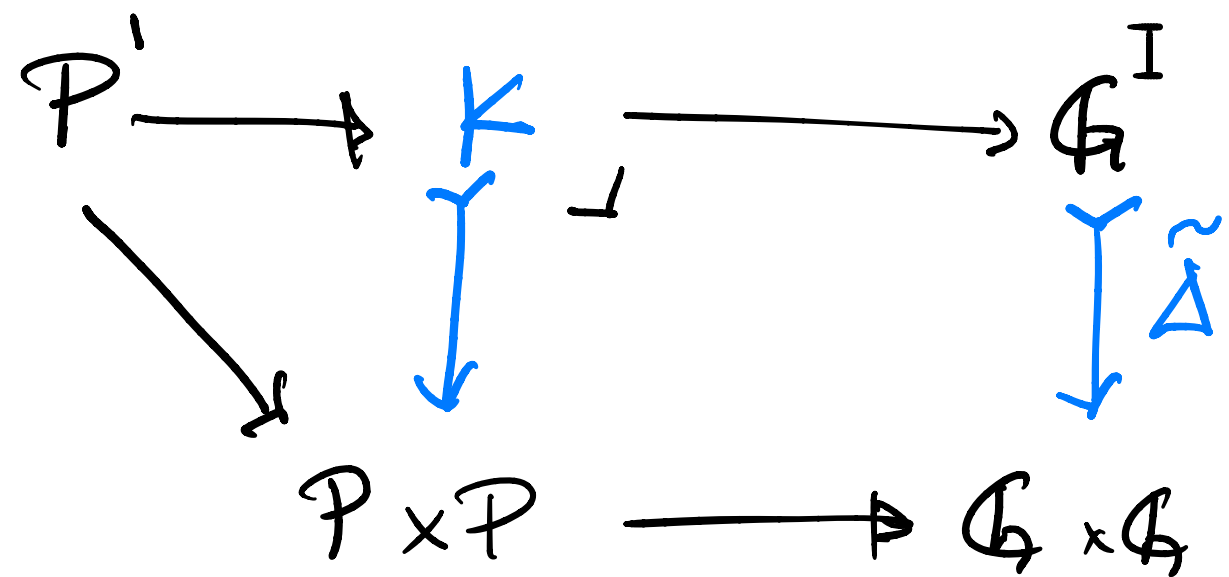


Prop. If \mathbb{G} is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since \mathbb{G} is cpt it has a cover $P \rightarrow \mathbb{G}$.

Take the p.b. \mathbb{K} and its cover P' , since Δ cpt.

Since \mathbb{G} is a 0-type,
 $\tilde{\Delta}: \mathbb{G}^I \rightarrow \mathbb{G} \times \mathbb{G}$ is monic.



Hence so is $K \rightarrow P \times P$,
 so K is an assembly.

Prop. If \mathbb{G} is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since \mathbb{G} is cpt it has a cover $P \rightarrow \mathbb{G}$.

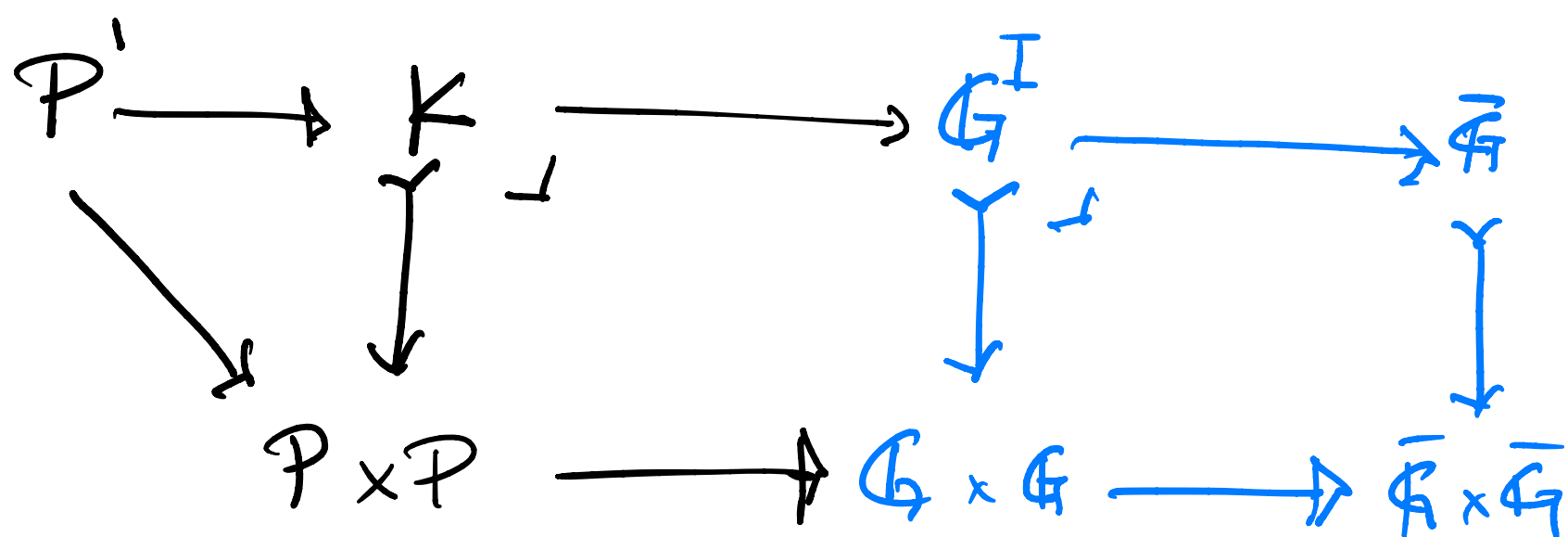
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Since \mathbb{G} is a 0-type

$\Delta: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ is monic.

Hence so is $\mathbb{K} \rightarrow P \times P$,

so \mathbb{K} is an assembly.



Write $\bar{\mathbb{G}} = \pi_0 \mathbb{G} = G_0/G_1$.

Prop. If \mathbb{G} is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since G_0 is cpt it has a cover $P \rightarrow \mathbb{G}$.

Take the p.b. \mathbb{K} and its cover P' .

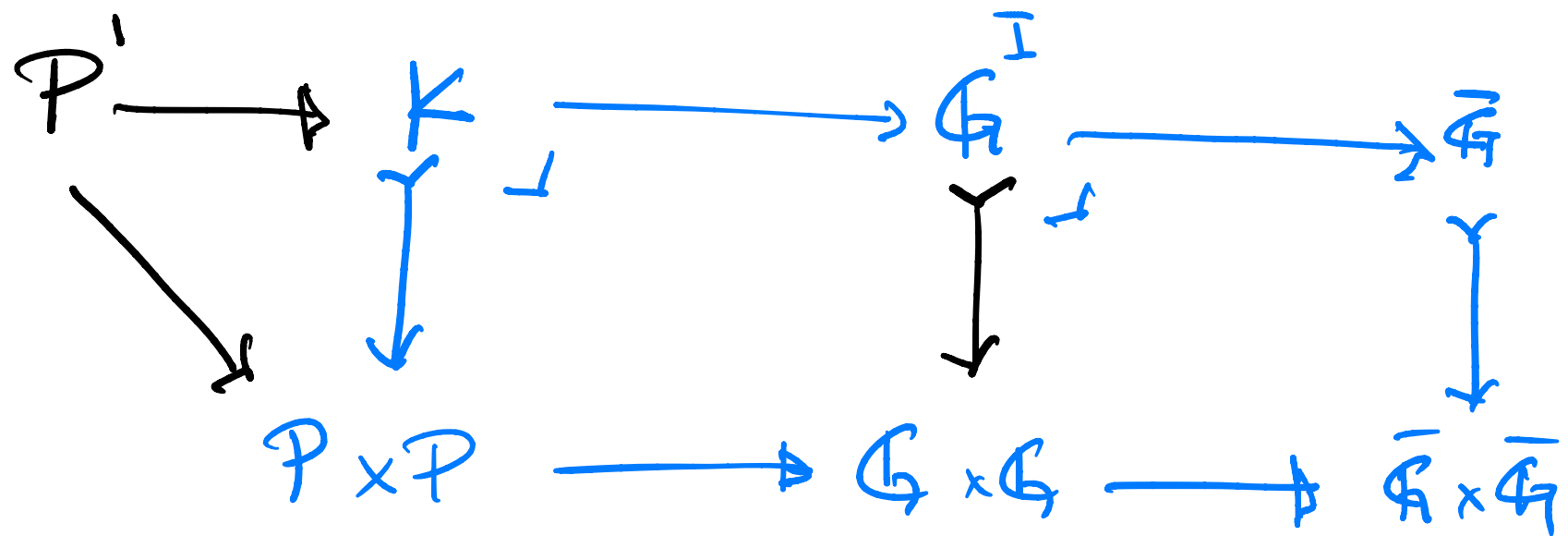
Since \mathbb{G} is a 0-type

$\mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ is monic.

Hence so is $\mathbb{K} \rightarrow P \times P$,

so \mathbb{K} is an assembly.

Write $\bar{\mathbb{G}} = \pi_0 \mathbb{G} = G_0/G_1$.



We then have $\mathbb{K} \rightrightarrows P \twoheadrightarrow \bar{\mathbb{G}}$ exact.

Prop. If \mathbb{G} is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since \mathbb{G} is cpt it has a cover $P \rightarrow \mathbb{G}$.

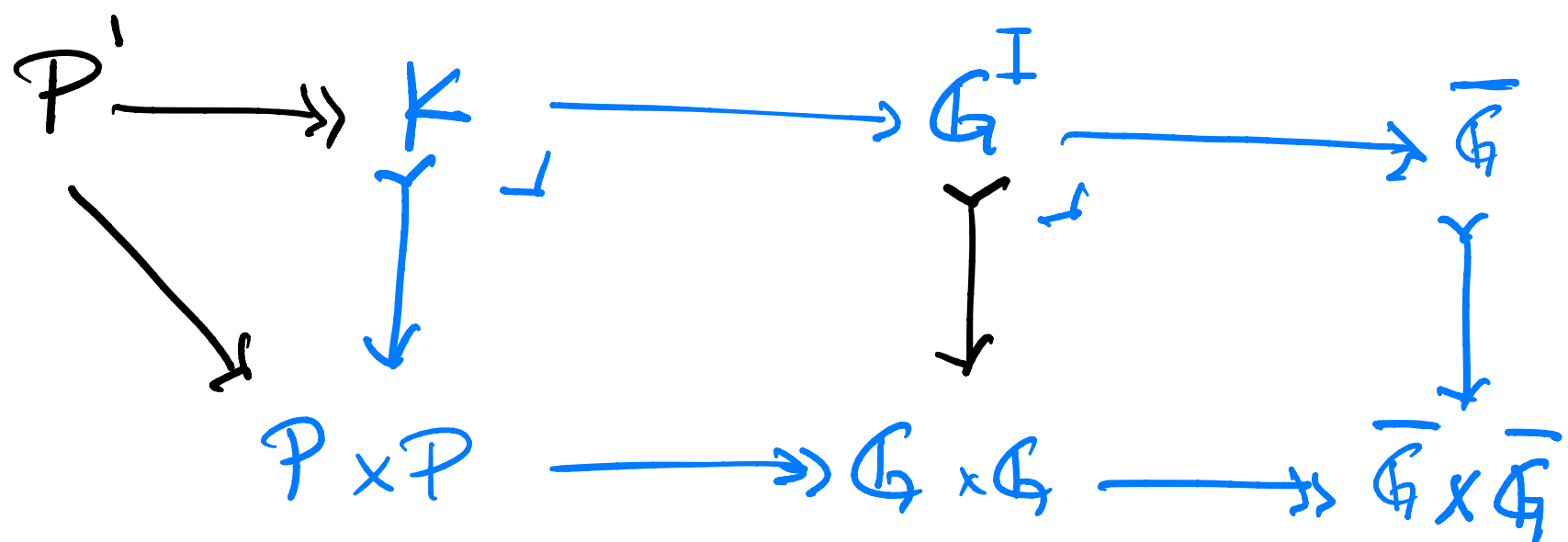
Take the p.b. \mathbb{K} and its cover P' .

Since \mathbb{G} is a 0-type
 $\mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ is monic.

Hence so is $\mathbb{K} \rightarrow P \times P$,
 so \mathbb{K} is an assembly.

Write $\overline{\mathbb{G}} = \pi_0 \mathbb{G} = \mathbb{G}_0 / \mathbb{G}_1$.

We then have $\mathbb{K} \rightrightarrows P \twoheadrightarrow \overline{\mathbb{G}}$ exact.



So $\overline{\mathbb{G}} = \pi_0 \mathbb{G}$ is coherent.

Prop. If \mathbb{G} is a coherent 0-type, then $\pi_0 \mathbb{G}$ is coherent.

Pf. Since \mathbb{G}_0 is cpt it has a cover $P \rightarrow \mathbb{G}$.

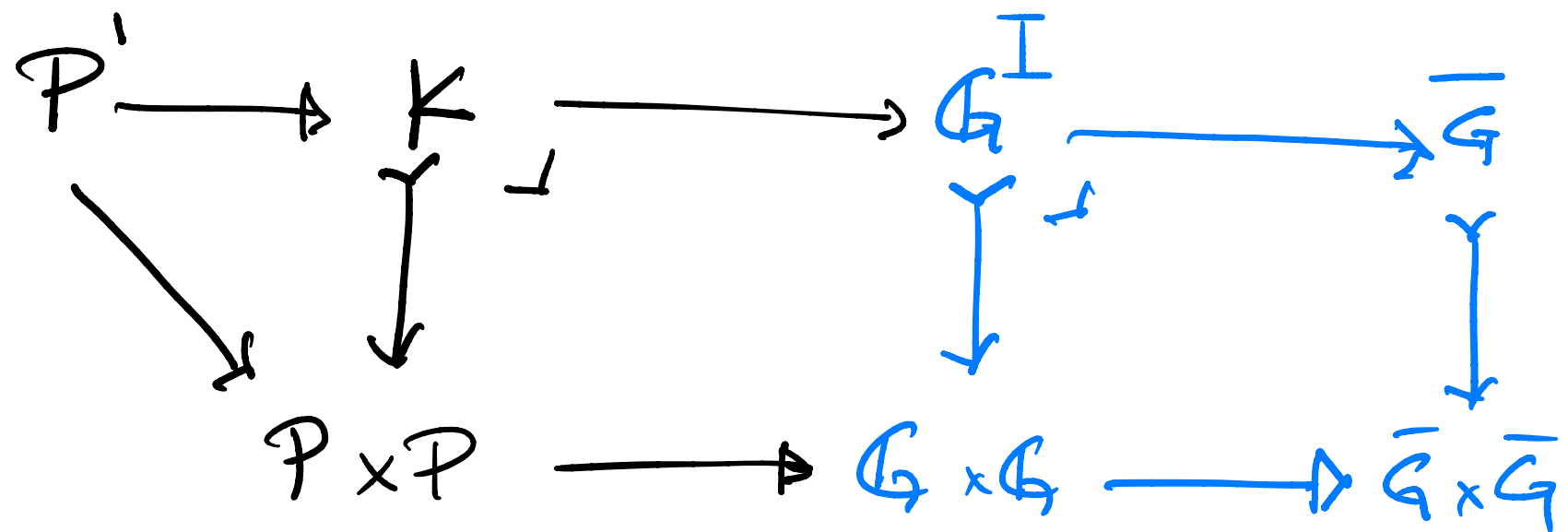
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Write $\bar{\mathbb{G}} = \pi_0 \mathbb{G} = \mathbb{G}_0 / \mathbb{G}_1$.

We then have $\mathbb{K} \rightrightarrows P \twoheadrightarrow \bar{\mathbb{G}}$ exact.

So $\bar{\mathbb{G}} = \pi_0 \mathbb{G}$ is coherent.

Finally, $\mathbb{G} \simeq \bar{\mathbb{G}}$ is a weak equivalence. \square

Thm Over $\hat{\mathbb{P}}$, we have

$$\begin{array}{ccc} \text{Eff} & \cong & \text{CohGpd}_0 \longrightarrow \text{Gpd}_0 \\ & & \downarrow \quad \downarrow \\ & & \text{CohGpd} \longrightarrow \text{Gpd} \end{array}$$

Then Over $\hat{\mathbb{P}}$, we have

$$\mathcal{E}ff \simeq \text{CohGpd}_0 \longrightarrow \text{Gpd}_0$$

Take $\mathcal{E}ff^2 := \text{CohGpd} \longrightarrow \text{Gpd}$.

to get an elementary $(2,1)$ -topos with

$$\mathcal{E}ff_0^2 \simeq \mathcal{E}ff .$$

THANKS !