Title: Free bicompletions of categories revisited

Speaker: André Joyal

Abstract: We revisit the theory of free bicompletion of categories. Category theory was extended to enriched ∞-categories by many authors (J. Lurie [2017], D. Gepner and R. Haugseng [2015] H. Heine [2024]). Let $\mathcal{V}$ a locally small bicomplete symmetric monoidal closed ∞-category and $\mathcal{V}\mathsf{CAT}$ be the category of $\mathcal{V}$-categories. A $\mathcal{V}$-*weight* is defined to be $\mathcal{V}$-diagram $W : \mathcal{J} \to \mathcal{V}$, and a $\mathcal{V}$-*scale* to be a class α of $\mathcal{V}$-weights; a $\mathcal{V}$-category $\mathcal{E}$ is α-*complete* if it is closed under weighted limits $\{W, -\}$ for all $W \in \alpha$. Dually, a $\mathcal{V}$-category $\mathcal{E}$ is α-*cocomplete* if the opposite category $\mathcal{V}$-category $\mathcal{E}^{op}$ is α-complete. If (α, β) is a pair of scales, a $\mathcal{V}$-category $\mathcal{E}$ is (α, β)-*bicomplete* if it is α-cocomplete and β-complete. We show that every $\mathcal{V}$-category $\mathcal{C}$ generates freely an (α, β)-bicomplete $\mathcal{V}$-category $\lambda : \mathcal{C} \to {}^{\alpha}\Lambda^{\beta}(\mathcal{C})$, moreover, it can be characterised by the following properties:

1. the functor $\lambda : \mathcal{C} \to {}^{\alpha}\Lambda^{\beta}(\mathcal{C})$ is fully faithful;
2. the category ${}^{\alpha}\Lambda^{\beta}(\mathcal{C})$ is (α, β)-bicomplete and generated by $\lambda(\mathcal{C})$;
3. every object in $\lambda(\mathcal{C})$ is an (α, β)-atoms;
4. the category ${}^{\alpha}\Lambda^{\beta}(\mathcal{C})$ is (α, β)-*soft*.

The general theory similar to the case where $\mathcal{V} = \mathsf{Set}$.

If time permits, I will propose a *philosophical* interpretation of these results.